

COMMUTING SYMPLECTOMORPHISMS AND DEHN TWISTS IN DIVISORS

DMITRY TONKONOG

ABSTRACT. Let X be a symplectic manifold satisfying a condition slightly stronger than weak monotonicity, and $f, g : X \rightarrow X$ be two commuting symplectomorphisms. We define an action of f on the Floer homology $HF(g)$ and an action of g on $HF(f)$, and prove the supertraces of these actions are equal. Using this, we obtain a topological lower bound on $\dim HF(g)$ if $g : X \rightarrow X$ is a symplectomorphism commuting with a symplectic involution on X . We apply this bound to the following setting.

Let $X \subset Gr(k, n)$ be a smooth divisor of degree within $[3; n]$ or $[k(n - k) + n - 2; +\infty)$. Let $L \subset X$ be Lagrangian sphere which is a vanishing cycle for an algebraic degeneration of X . It defines a symplectomorphism $\tau_L : X \rightarrow X$ called the Dehn twist. We prove that τ_L gives an element of infinite order in the group of symplectomorphisms of X modulo Hamiltonian isotopy.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Overview. Let X be a symplectic manifold and $\text{Symp}(X)/\text{Ham}(X)$ be the group of all symplectomorphisms of X modulo Hamiltonian isotopy. When X is simply-connected, this group is the same as $\pi_0\text{Symp}(X)$. If one denotes by $\pi_0\text{Diff}(X)$ the smooth mapping class group, there is an obvious forgetful map

$$\text{Symp}(X)/\text{Ham}(X) \xrightarrow{\text{forgetful}} \pi_0\text{Diff}(X).$$

Paul Seidel in his thesis [32] found examples when this map is not injective: if X is any complete intersection of complex dimension 2 other than \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$ and $\tau : X \rightarrow X$ is a certain symplectomorphism called the Dehn twist, then τ^2 is smoothly isotopic to the identity, but not Hamiltonian isotopic to the identity. Later Seidel proved [34] that the kernel of the above map is infinite for some K3 surfaces X , again by considering the group generated by a Dehn twist. Using a new technique, we study Dehn twists in certain divisors (the main example are divisors in Grassmannians) and extend the range of examples when the above forgetful map has infinite kernel.

Suppose X satisfies the so-called W^+ condition which is slightly stronger than weak monotonicity. We define, for two commuting symplectomorphisms $f, g : X \rightarrow X$, their actions on Floer homology $f_{\text{floer}} : HF(g) \rightarrow HF(g)$, $g_{\text{floer}} : HF(f) \rightarrow HF(f)$. We then prove a theorem which was proposed by Paul Seidel, cf. [31, Remark 4.1], who suggested to call it the elliptic relation.

Theorem 1.1 (Elliptic relation). *If X is a symplectic manifold satisfying the W^+ condition and $f, g : X \rightarrow X$ are two commuting symplectomorphisms, then*

$$STr(f_{\text{floer}}) = STr(g_{\text{floer}}) \in \Lambda.$$

Here Λ is the Novikov field. In the rest of the introduction, we explain the elliptic relation (and also state its Lagrangian version) together with its application to Dehn twists in divisors, starting from the latter.

1.2. Order of Dehn twists in divisors. Let $Gr(k, n)$ be the Grassmannian of k -planes in \mathbb{C}^n . Let $\mathcal{O}(d)$ be the line bundle on $Gr(k, n)$ which is the pullback of $\mathcal{O}_{\mathbb{P}^N}(d)$ under the Plücker embedding $Gr(k, n) \subset \mathbb{P}^N$. Consider a smooth divisor $X \subset Gr(k, n)$ in the linear system $|\mathcal{O}(d)| = \mathbb{P}H^0(Gr(k, n), \mathcal{O}(d))$. The results below are interesting even for $\mathbb{P}^{n-1} = Gr(1, n)$, so for simplicity one can take $X \subset \mathbb{P}^{n-1}$ to be a smooth projective hypersurface of degree d throughout this subsection.

For $d \geq 2$, X contains a class of Lagrangian spheres which we call $|\mathcal{O}(d)|$ -vanishing Lagrangian spheres, see Definition 3.7. Briefly, these spheres are vanishing cycles for algebraic degenerations of X inside the linear system $|\mathcal{O}(d)|$. To every parametrised Lagrangian sphere $L \subset X$ one associates a symplectomorphism $\tau_L : X \rightarrow X$ called the Dehn twist around L , see Definition 3.11. We prove the following.

Theorem 1.2. *Let $X \subset Gr(k, n)$ be a smooth divisor in the linear system $|\mathcal{O}(d)|$, and $L \subset X$ be an $|\mathcal{O}(d)|$ -vanishing Lagrangian sphere. Suppose*

$$3 \leq d \leq n \quad \text{or} \quad d \geq k(n - k) + n - 2.$$

Then the Hamiltonian isotopy class of τ_L is an element of infinite order in the group $\text{Symp}(X)/\text{Ham}(X)$.

When $d = 2$ and $k = 1$ (X is a projective quadric), τ_L has order 1 or 2 depending on the parity of n [40, Lemma 4.2]. While our proof crucially uses $d \geq 3$, further restrictions on d are only needed to make X satisfy the W^+ condition, so that the ‘classical’ definition of Floer homology of symplectomorphisms $X \rightarrow X$ applies. There are techniques [13] defining Floer homology of symplectomorphisms on arbitrary symplectic manifolds. With their help the proof of Theorem 1.2 (and of Theorem 1.1) should work for all $d \geq 3$.

Recall the forgetful map $\text{Symp}(X)/\text{Ham}(X) \rightarrow \pi_0 \text{Diff}(X)$. If $\dim X = k(n - k) - 1$ is odd and $d \geq 3$, the image of τ_L has infinite order in $\pi_0 \text{Diff}(X)$ by the Picard-Lefschetz formula, so Theorem 1.2 becomes trivial. However, when $\dim X$ is even the image of τ_L has finite order in $\pi_0 \text{Diff}(X)$, see Subsection 3.4 for details, so Theorem 1.2 is really of symplectic nature. When X is Calabi-Yau ($d = n$), Theorem 1.2 follows from a grading argument of Paul Seidel [34]. Theorem 1.2 is new in all cases when $\dim X$ is even and $d \neq n$, for instance it appears to be new even for the cubic surface $X \subset \mathbb{P}^3$.

Let

$$\Delta \subset \mathbb{P}H^0(\text{Gr}(k, n), \mathcal{O}(d))$$

be the discriminant variety parameterising all singular divisors in $|\mathcal{O}(d)|$. Theorem 1.2 implies a corollary about the fundamental group of the complement to the discriminant. Fix a divisor $X \in |\mathcal{O}(d)|$. For any family $X_t \subset \text{Gr}(k, n)$ of smooth divisors in $|\mathcal{O}(d)|$, $t \in [0; 1]$, there is a symplectic parallel transport map, a symplectomorphism $X_0 \rightarrow X_1$, which depends up to Hamiltonian isotopy only on the homotopy class of the path X_t relative to its endpoints. Applied to loops, parallel transport gives the symplectic monodromy map

$$\pi_1(\mathbb{P}H^0(\text{Gr}(k, n), \mathcal{O}(d)) \setminus \Delta) \xrightarrow{\text{monodromy}} \text{Symp}(X)/\text{Ham}(X).$$

The discriminant complement contains a distinguished conjugacy class of loops γ called meridian. A meridian loop

$$\gamma \subset \mathbb{P}H^0(\text{Gr}(k, n), \mathcal{O}(d)) \setminus \Delta$$

is the boundary of a 2-disk in $\mathbb{P}H^0(\text{Gr}(n, k), \mathcal{O}(d))$ that intersects Δ transversely once. The image of such a loop under the monodromy map is the Dehn twist τ_L where $L \subset X$ is an $|\mathcal{O}(d)|$ -vanishing Lagrangian sphere. Theorem 1.2 implies the following.

Corollary 1.3. *If $3 \leq d \leq n$ or $d \geq k(n - k) + n - 2$ and $\gamma \subset \mathbb{P}H^0(\text{Gr}(k, n), \mathcal{O}(d)) \setminus \Delta$ is a meridian loop, then*

$$[\gamma] \in \pi_1(\mathbb{P}H^0(\text{Gr}(k, n), \mathcal{O}(d)) \setminus \Delta) \quad \text{is an element of infinite order.}$$

Note that $[\gamma] \in H_1(\mathbb{P}H^0(\text{Gr}(k, n), \mathcal{O}(d)) \setminus \Delta; \mathbb{Z})$ has finite order. For the projective space $\text{Gr}(1, n) = \mathbb{P}^{n-1}$, the fundamental group $\pi_1(\mathbb{P}H^0(\mathbb{P}^{n-1}, \mathcal{O}(d)) \setminus \Delta)$ is computed by Michael Lönne in [20]. That computation implies Corollary 1.3 for $k = 1$. For $k \neq 1$, the corresponding fundamental group seems not to be studied and Corollary 1.3 is new, except for the cases when $k(n - k)$ is even or $d = n$ as discussed above.

We prove analogues of Theorem 1.2 and Corollary 1.3 for divisors of some very ample line bundles $\mathcal{L} \rightarrow Y$, where Y is a Kähler manifold which carries a holomorphic involution with certain properties. A precise statement is postponed to Subsection 1.7.

1.3. Elliptic relation for commuting symplectomorphisms. To prove Theorem 1.2, we use the elliptic relation (Theorem 1.1) which we now discuss.

Let X be a symplectic manifold satisfying the W^+ condition (e.g. X is Kähler and Fano or K_X is sufficiently positive), see Definition 2.1. For any symplectomorphism $f : X \rightarrow X$ one defines its Floer cohomology $HF(f)$. It is a \mathbb{Z}_2 -graded vector space, $HF(f) = HF^0(f) \oplus HF^1(f)$, over the Novikov field

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i q^{\omega_i} : a_i \in \mathbb{C}, \omega_i \in \mathbb{R}, \lim_{i \rightarrow \infty} \omega_i = +\infty \right\}.$$

For any two commuting symplectomorphisms $f, g : X \rightarrow X$ we define invertible automorphisms

$$g_{\text{floer}} : HF(f) \rightarrow HF(f) \quad \text{and} \quad f_{\text{floer}} : HF(g) \rightarrow HF(g).$$

The definition briefly is that we pick a time-dependent almost complex structure J and a Hamiltonian H to define the group $HF(f; J, H)$. It is canonically isomorphic (on the chain level) to $HF(gfg^{-1}; g^*J, H \circ g)$ by composing all pseudo-holomorphic curves with g . If f, g commute, g_{floer} is the composition of isomorphisms

$$HF(f; J, H) \longrightarrow HF(gfg^{-1}; g^*J, H \circ g) = HF(f; g^*J, H \circ g) \longrightarrow HF(f; J, H)$$

where the last arrow is the continuation map associated to a homotopy of data from $(g^*J, H \circ g)$ to (J, H) .

The automorphisms $f_{\text{floer}}, g_{\text{floer}}$ have zero degree, and one can define their supertrace:

$$STr(g_{\text{floer}}) := Tr(g_{\text{floer}}|_{HF^0(f)}) - Tr(g_{\text{floer}}|_{HF^1(f)}) \in \Lambda.$$

Recall that Theorem 1.1 asserts that $STr(f_{\text{floer}}) = STr(g_{\text{floer}})$.

Now suppose a symplectomorphism f commutes with a finite-order symplectomorphism ϕ , $\phi^k = \text{Id}$, with smooth fixed locus X^ϕ . Using an argument reminiscent of the PSS isomorphism, we show that

$$STr(f_{\text{floer}} : HF(\phi) \rightarrow HF(\phi)) = L(f|_{X^\phi}) \cdot q^0.$$

The right hand side is the topological Lefschetz number

$$L(f|_{X^\phi}) = Tr(f^*|_{H^{\text{even}}(X^\phi)}) - Tr(f^*|_{H^{\text{odd}}(X^\phi)})$$

where $f^* : H^*(X^\phi) \rightarrow H^*(X^\phi)$ is the classical action on the cohomology of X^ϕ . On the other hand, using that ϕ has finite order, we show that $STr(\phi_{\text{floer}} : HF(f) \rightarrow HF(f))$ equals $a \cdot q^0$ where $|a| \leq \dim_\Lambda HF(f)$. Combining this with the elliptic relation we obtain the following corollary.

Proposition 1.4. *Let X be a symplectic manifold satisfying the W^+ condition, $f, \phi : X \rightarrow X$ two commuting symplectomorphisms and $\phi^k = \text{Id}$. Suppose the fixed locus $X^\phi \subset X$ is a smooth manifold. Then*

$$\dim_\Lambda HF(f) \geq |L(f|_{X^\phi})|.$$

Remark 1.5. The fixed locus X^ϕ is allowed to be disconnected, with components of different dimensions.

Remark 1.6. If $f : X \rightarrow X$ is a diffeomorphism with smooth fixed locus X^f such that $\text{Id} - df(x)|_{N_x \Sigma}$ is non-degenerate on the normal space $N_x \Sigma \subset T_x X$ to any connected component $\Sigma \subset X^f$ for every $x \in \Sigma$, then $L(f) = \sum_{\Sigma \subset X^f} \text{sign det}(\text{Id} - df|_{N_x \Sigma}) \cdot \chi(\Sigma)$. Consequently, if $\phi, \psi : X \rightarrow X$ are non-degenerate finite order symplectomorphisms, we get $L(\phi|_{X^\psi}) = L(\psi|_{X^\phi}) = \chi(X^\phi \cap X^\psi)$, provided the latter intersection is clean. This agrees with the elliptic relation and the topological interpretation of the Floer-homological actions for finite order maps.

Remark 1.7. There is a more straightforward proof of Proposition 1.4 which does not appeal to Theorem 1.1, but still requires some analysis in the spirit of [36, Lemma 14.11]. See Remark 2.22 for more details.

Remark 1.8. Theorem 1.1 holds when f, g commute only up to Hamiltonian isotopy, and more generally when fg^{-1} is isomorphic to Id in the Donaldson category whose objects are symplectomorphisms of X and $\text{Hom}(f, g) = HF(fg^{-1})$; the proofs require only minor modifications. On the other hand, we cannot weaken the condition that f, ϕ strictly commute in Proposition 1.4.

1.4. Outline of proof of Theorem 1.1. This sketch is intended for readers familiar with Floer theory. The necessary definitions and the complete proof are found in Section 2. Figure 1 illustrates the argument.

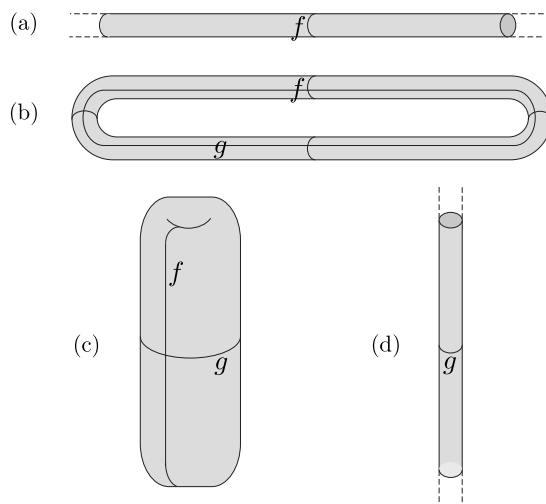


FIGURE 1. Changing the base of a symplectic fibration in the proof of Theorem 1.1.

Let f, g be two commuting symplectomorphisms. By our definition, the supertrace $STr(g_{\text{floer}})$ is computed by counting certain solutions to Floer's continuation equation, equivalently by counting holomorphic sections of a certain symplectic fibration $E_f \rightarrow S^1 \times \mathbb{R}$, see Figure 1(a). This fibration has monodromy f along S^1 , and the almost complex structure on E_f differs by the action of g over the two ends of the cylinder. We count only those sections whose asymptotics differ by the action of g at the ends of the cylinder. One can therefore glue the fibration, together with the almost complex structure, into a fibration $E_{f,g} \rightarrow S^1 \times S^1$. A gluing theorem in Symplectic Field Theory gives a bijection between holomorphic sections $S^1 \times \mathbb{R} \rightarrow E_f$ (with asymptotics as above) and all holomorphic sections $S^1 \times S^1 \rightarrow E_{f,g}$ where $S^1 \times S^1$ is endowed with the complex structure very 'long' in the direction of the second S^1 -factor: see Figure 1(b). We will refer to the mentioned bijection by $(*)$ in the next few paragraphs.

On the other hand, the count of holomorphic sections $S^1 \times S^1 \rightarrow E_{f,g}$ does not depend on the chosen complex structure on $S^1 \times S^1$. Take another complex structure on $S^1 \times S^1$ which is 'long' in the first S^1 -factor instead of the second one, see Figure 1(c). The same gluing argument as above $(*)$ implies that the count of holomorphic sections $S^1 \times S^1 \rightarrow E_{f,g}$ equals to the count of holomorphic sections $\mathbb{R} \times S^1 \rightarrow E_g$ (with asymptotics different by f over the ends of the cylinder), where $E_g \rightarrow \mathbb{R} \times S^1$ is the fibration obtained by cutting $E_{f,g}$ along the first S^1 -factor, see Figure 1(d). Similarly to what we began with, the count of holomorphic sections $\mathbb{R} \times S^1 \rightarrow E_g$ is equal to $STr(f_{\text{floer}})$.

The key difficulty in upgrading this sketch to a proof is to determine how the bijection $(*)$ behaves with respect to the signs attached to sections over the cylinder (which in general depend on the choice of a 'coherent orientation' but are canonical for sections contributing to the supertrace) and signs canonically attached to sections over the torus. The outcome is that $(*)$ multiplies signs by $(-1)^{\deg x}$ where x is a $\pm\infty$ asymptotic periodic orbit of the section over the cylinder. (The $\pm\infty$ asymptotics differ by g and have the same degree.) This is formula (2.25) in Section 2, it explains why Theorem 1.1 is an equality between supertraces and not usual traces. We have not found a suitable reference for the sign formula (2.25). (Coherent orientations in SFT are discussed in [10, 7], see especially [7, Corollary 7], but don't seem to give the result we need).

1.5. Elliptic relation for invariant Lagrangians. Before explaining how the elliptic relation helps to prove Theorem 1.2, let us discuss its Lagrangian version. The coefficient field is still Λ . Definitions and sketch proofs are briefly presented in Subsection 2.13.

Let X be a connected monotone symplectic manifold (e.g. complex Fano variety), and $L_1, L_2 \subset X$ monotone Lagrangians (e.g. simply connected). Suppose there is a symplectomorphism $\phi : X \rightarrow X$ such that $\phi(L_1) = L_1$, $\phi(L_2) = L_2$. Under a condition involving spin structures, a version of the open-closed string map provides twisted homology classes $[L_1]^\phi \in HF(\phi)$, $[L_2]^{\phi^{-1}} \in HF(\phi^{-1})$. (We consider of homology, not cohomology, in this subsection.) Consider the quantum product $[L_1]^\phi * [L_2]^{\phi^{-1}} \in QH_*(X)$ and the counit $\chi : QH_*(X) \rightarrow \Lambda$ (it computes the coefficient by

the generator of $QH_0(X) \cong \Lambda$). Under some extra assumptions appearing in the next theorem, there is again an action $\phi_{\text{floer}} : HF(L_1, L_2) \rightarrow HF(L_1, L_2)$.

Theorem 1.9 (Elliptic relation). *Suppose (X, L_1, L_2) are monotone, $\phi : X \rightarrow X$ is a symplectomorphism, $\phi(L_i) = L_i$. If the base field has $\text{char} \neq 2$, suppose L_i are orientable and there is a special choice of spin structures on L_i described in Subsection 2.13 (it exists if L_i are simply-connected). Then*

$$STr(\phi_{\text{floer}}) = \chi \left([L_1]^\phi * [L_2]^{\phi^{-1}} \right).$$

If $\phi^k = \text{Id}$ and the fixed loci $L_i^\phi \subset X^\phi$ are smooth and orientable, the q^0 -term of the right hand side equals the classical homological intersection $[L_1^\phi] \cdot [L_2^\phi] \in \mathbb{Z}$ inside X^ϕ , where $[L_i^\phi] \in H_{\dim_{\mathbb{R}} X/2}(X; \mathbb{Z})$. On the other hand, eigenvalue decomposition of ϕ_{floer} implies that the left hand side equals $a \cdot q^0$ with $a \in \mathbb{C}$, $|a| \leq \dim_{\Lambda} HF(L_1, L_2)$. The elliptic relation yields the following analogue of Proposition 1.4.

Proposition 1.10. *Under the assumptions of Theorem 1.9, if $\phi^k = \text{Id}$ and the fixed loci L_i^ϕ, X^ϕ are smooth and orientable then*

$$\dim_{\Lambda} HF(L_1, L_2) \geq \left| [L_1^\phi] \cdot [L_2^\phi] \right|.$$

In the monotone setting, one can pass from Λ -coefficients to the base field (e.g. \mathbb{C} or $\mathbb{Z}/2\mathbb{Z}$) without changing dimensions of Floer homology [42, Remark 4.4]. So Proposition 1.10 gives the same bound on $\dim HF(L_1, L_2; \mathbb{C})$ or $\dim HF(L_1, L_2; \mathbb{Z}/2\mathbb{Z})$. However, the proof of Proposition 1.10 crucially uses Theorem 1.9 over Λ , as can be seen from the sketch we presented.

As a simple application of Proposition 1.10 we can recover the following known fact: $\mathbb{R}\mathbb{P}^n \subset \mathbb{C}\mathbb{P}^n$ is not self-displaceable by a Hamiltonian isotopy, as $\dim HF(\mathbb{R}\mathbb{P}^n, \mathbb{R}\mathbb{P}^n; \mathbb{Z}/2\mathbb{Z}) \geq 1$. When n is even, this is true because the Euler characteristic of $\mathbb{R}\mathbb{P}^n$ equals 1; when n is odd, consider the hyperplane reflection ι on $\mathbb{C}\mathbb{P}^n$ so that $(\mathbb{R}\mathbb{P}^n)^\iota = \mathbb{R}\mathbb{P}^{n-1}$ and apply Proposition 1.10.

In Appendix A we provide a more interesting application of Proposition 1.10. Namely, we prove that for $L \subset X$ as in Theorem 1.2, and if X is in addition Fano and even-dimensional, there is an isomorphism of rings $HF(L, L; \mathbb{C}) \cong \mathbb{C}[x]/x^2$.

Remark 1.11. The action ϕ_{floer} on $HF(L_1, L_2)$ (as well the actions in the case of two commuting symplectomorphisms) can be defined using functors coming from Lagrangian correspondences [43, 44]. It is possible that the two versions of the elliptic relation admit a generalisation for Lagrangian correspondences. Our results could also be related to categorical Lefschetz-type formulas studied in [25, 4].

1.6. Outline of proof of Theorem 1.2. We have already mentioned that Theorem 1.2 holds for topological reasons when $\dim X$ is odd. Suppose therefore that $\dim_{\mathbb{C}} Gr(k, n)$ is odd. The Grassmannian has an involution ι whose fixed locus contains an even-dimensional connected component $\tilde{\Sigma} \subset Gr(k, n)$. For example, when $k = 1$ we can take the involution $(x_1 : x_2 : x_3 : x_4 : \dots : x_n) \mapsto (-x_1 : -x_2 : -x_3 : x_4 : \dots : x_n)$ and $\tilde{\Sigma} = \mathbb{P}^2(x_1 : x_2 : x_3)$.

The key idea of reducing Theorem 1.2 to Proposition 1.4 is the following construction performed in Proposition 4.2. We construct a smooth divisor $X \subset Gr(k, n)$ invariant under ι such that the fixed locus X^ι of the involution $\iota|_X$ contains an odd-dimensional connected component $\Sigma = \tilde{\Sigma} \cap X$. Next, we construct two ι -invariant $|\mathcal{O}(d)|$ -vanishing Lagrangian spheres $L_1, L_2 \subset X$ which intersect each other transversely once. Moreover, the fixed loci $L_i^\iota := L_i \cap \Sigma$, $i = 1, 2$, are Lagrangian spheres in Σ which intersect each other transversely once, see Figure 2. This is where we need $d \geq 3$.

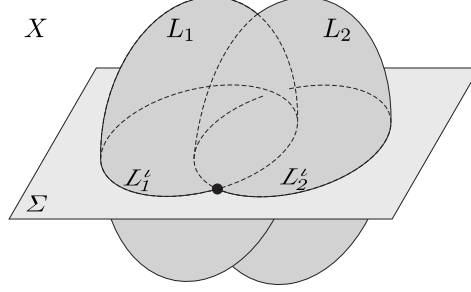


FIGURE 2. Invariant Lagrangian spheres L_1 and L_2 used in the proof of Theorem 1.2.

Theorem 1.2 is proved in Section 5. Consider the product of iterated Dehn twists $\tau_{L_1}^{2k} \tau_{L_2}^{2k}$. Because L_1, L_2 are ι -invariant, $\tau_{L_1}^{2k} \tau_{L_2}^{2k}$ can be made ι -equivariant. The Lefschetz number of $(\tau_{L_1}^{2k} \tau_{L_2}^{2k})|_\Sigma = \tau_{L_1^\iota}^{2k} \tau_{L_2^\iota}^{2k}$ on $H^*(\Sigma)$ is equal to $c - 4k^2$, where c is a constant. This follows from the Picard–Lefschetz formula and crucially uses the fact $\dim \Sigma$ is odd. If $\dim \Sigma$ were even, the trace would be independent of k .

Consequently by Proposition 1.4, $\dim HF(\tau_{L_1}^{2k} \tau_{L_2}^{2k})$ grows with k . So $\tau_{L_1}^{2k} \tau_{L_2}^{2k}$ is not Hamiltonian isotopic to Id if $k \neq 0$.

Finally we note that L_1, L_2 from our construction can be taken one to another by a symplectomorphism of X . This means τ_{L_1} and τ_{L_2} are conjugate. If $\tau_{L_1}^{2k}$ was Hamiltonian isotopic to Id , then so would be $\tau_{L_2}^{2k}$ and the product $\tau_{L_1}^{2k} \tau_{L_2}^{2k}$. Consequently, $\tau_{L_1}^{2k}$ is not Hamiltonian isotopic to Id for $k \neq 0$, hence τ_{L_1} has infinite order in $\text{Symp}(X)/\text{Ham}(X)$. Theorem 1.2 is proved for the specially constructed $|\mathcal{O}(d)|$ -vanishing Lagrangian sphere $L_1 \subset X$. If X' is another smooth divisor linearly equivalent to X and $L' \subset X'$ is another $|\mathcal{O}(d)|$ -vanishing Lagrangian sphere, Lemma 3.8 says there is a symplectomorphism $X \rightarrow X'$ taking L to L' . It implies Theorem 1.2 in general.

1.7. A generalisation of Theorem 1.2. Theorem 1.2 is a particular case of the more general, but also more technical Theorem 1.12 which we now state. Let \mathcal{L} be a very ample line bundle over a Kähler manifold Y . It gives an embedding $Y \subset \mathbb{P}^N := \mathbb{P}H^0(Y, \mathcal{L})^*$.

Suppose $\iota : Y \rightarrow Y$ is a holomorphic involution which lifts to an automorphism of \mathcal{L} . So ι induces a linear involution on $H^0(Y, \mathcal{L})^*$, splitting it into the direct sum

of the ± 1 eigenspaces $H^0(Y, \mathcal{L})_{\pm}^*$. Let $\Pi_{\pm} \subset \mathbb{P}^N$ be the projectivisations of these eigenspaces. The fixed locus $Y^{\iota} \subset Y$ of the involution $\iota : Y \rightarrow Y$ is:

$$Y^{\iota} = (\Pi_+ \sqcup \Pi_-) \cap Y,$$

where the intersection is taken inside \mathbb{P}^N . It can have many connected components because the intersections $\Pi_+ \cap Y$, $\Pi_- \cap Y$ may be disconnected.

Theorem 1.12. *Let \mathcal{L} be a very ample line bundle over a Kähler manifold Y , and $\iota : Y \rightarrow Y$ a holomorphic involution which lifts to an automorphism of \mathcal{L} . Suppose the fixed locus Y^{ι} is smooth (maybe disconnected with components of different dimensions), and ι is non-degenerate (i.e. acts by $-\text{Id}$ on the normal bundle to Y^{ι}). Fix $d \geq 3$.*

Let $H^0(Y, \mathcal{L}^{\otimes d})_{\pm}$ denote the ± 1 -eigenspace of the involution on $H^0(Y, \mathcal{L}^{\otimes d})$ induced by ι . Let Π_{\pm} be as above. Suppose one of the following:

(a) *d is even, and*

Y^{ι} contains a connected component $\tilde{\Sigma}$ such that $\dim_{\mathbb{C}} \tilde{\Sigma}$ is even;

(b) *d is odd,*

there is a smooth divisor in the linear system $\mathbb{P}H^0(Y, \mathcal{L}^{\otimes d})_+$, and

$\Pi_+ \cap Y$ contains a connected component $\tilde{\Sigma}$ such that $\dim_{\mathbb{C}} \tilde{\Sigma}$ is even.

Let $X \subset Y$ be a smooth divisor in the linear system $|\mathcal{L}^{\otimes d}|$ and $L \subset X$ an $|\mathcal{L}^{\otimes d}|$ -vanishing Lagrangian sphere, see Definition 3.7. Denote by τ_L the Dehn twist around L ; it is a symplectomorphism of X , see Definition 3.11. Assume X satisfies the W^+ condition, see Definition 2.1.

Then the Hamiltonian isotopy class of τ_L is an element of infinite order in the group $\text{Symp}(X)/\text{Ham}(X)$.

The same is true if we replace symbols $+$ with symbols $-$ in Case (b).

Like Theorem 1.2, Theorem 1.12 is new when $\dim X$ is even and X is not Calabi-Yau.

In Case (a), the existence of a smooth ι -invariant divisor X follows from Bertini's theorem, so it is not included as a condition of the theorem. In Case (b), an invariant divisor can sometimes be found using a strong Bertini theorem [9, Corollary 2.4], which gives the following.

Lemma 1.13. *Under conditions of Theorem 1.12, let d be odd. There is a smooth divisor in the linear system $\mathbb{P}H^0(Y, \mathcal{L}^{\otimes d})_{\pm}$ if every connected component of $\Pi_{\mp} \cap Y$ has dimension less than $\frac{1}{2} \dim Y$.*

As in the beginning of the introduction, we have the following corollary.

Corollary 1.14. *Under conditions of Theorem 1.12, let $\gamma \subset \mathbb{P}H^0(Y, \mathcal{L}^{\otimes d}) \setminus \Delta$ be a meridian loop described in the beginning of the introduction. Then*

$$[\gamma] \in \pi_1 \left(\mathbb{P}H^0(Y, \mathcal{L}^{\otimes d}) \setminus \Delta \right) \quad \text{is an element of infinite order.}$$

We prove these statements in Section 5. We have earlier explained the plan of proof of Theorem 1.2; actually we follow this plan to prove the general Theorem 1.12 first, and then derive Theorem 1.2 from it.

1.8. Equivariant transversality approaches. This subsection is not used in the rest of the paper. Computations of Floer homology in the presence of a symplectic involution were discussed by Khovanov and Seidel [18], and Seidel and Smith [38]. Both of them imposed restrictive conditions on the involution which allow one to choose a regular equivariant almost complex structure for computing Floer homology.

In [38], it is proved that

$$\dim HF(L_1, L_2; \mathbb{Z}/2) \geq \dim HF(L_1^t, L_2^t; \mathbb{Z}/2)$$

when there exists a stable normal trivialization of the normal bundle to X^t respecting the L_i . In particular, the Chern classes of this normal bundle should vanish. The right-hand side is Floer homology inside X^t , where L_i^t are the fixed loci of L_i and X^t is the fixed locus of X . Sometimes the right-hand side is easier to compute than the left-hand side (e.g. when all intersection points $L_1^t \cap L_2^t$ have the same sign). However, the condition on the normal bundle makes this estimate inapplicable to divisors in $Gr(k, n)$.

In a very special case, [18] proves that

$$\dim HF(L_1, L_2; \mathbb{Z}/2) = |L_1^t \cap L_2^t|$$

where the right hand side is the unsigned count of intersection points. The assumption is, roughly, that the fixed locus X^t has real dimension 2 and $L_1^t, L_2^t \subset X^t$ are curves having minimal intersection in their homotopy class. One could prove a \mathbb{C} -version of this equality if L_i admit ι -equivariant Pin strictures, and apply it to divisors in $\mathbb{P}^{n-1} = Gr(1, n)$, i.e. projective hypersurfaces (thus giving an alternative proof of Theorem 1.2 in this case). However, it cannot be applied to divisors in general Grassmannians. When $k > 2$, $Gr(k, n)$ has no holomorphic involution with a connected component of complex dimension 2; this is easy to check because all holomorphic automorphisms $Gr(k, n)$ come from linear ones on \mathbb{C}^n , with a single exception when $n = 2k$ [8, Theorem 1.1 (Chow)].

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2. THE ELLIPTIC RELATION

This section proves the symplectomorphism elliptic relation (together with Proposition 1.4) and sketches a proof of its Lagrangian version.

2.1. Floer homology and continuation maps.

Definition 2.1 (The W^+ condition). A symplectic manifold (X, ω) satisfies the W^+ condition [33] if for every $A \in \pi_2(M)$

$$2 - n \leq c_1(A) \leq -1 \implies \omega(A) \leq 0.$$

Let (X, ω) be a compact symplectic manifold satisfying the W^+ condition. Fix a symplectomorphism $f : X \rightarrow X$. In this subsection we recall the definition of Floer homology $HF(f)$. Basic references are [22, 27, 33].

Take a family of ω -tame almost complex structures J_s on X , and a family of Hamiltonian functions $H_s : X \rightarrow \mathbb{R}$, $s \in \mathbb{R}$. They must be f -periodic:

$$(2.1) \quad H_s = H_{s+1} \circ f, \quad J_s = f^* J_{s+1}.$$

By X_{H_s} we denote the Hamiltonian vector field of H_s , and by $\psi_s : X \rightarrow X$ the Hamiltonian flow:

$$(2.2) \quad d\psi_s/ds = X_{H_s} \circ \psi_s, \quad \psi_0 = \text{Id}.$$

The following equation on $u(s, t) : \mathbb{R}^2 \rightarrow X$ is called Floer's equation:

$$(2.3) \quad \partial u / \partial t + J_s(u)(\partial u / \partial s - X_{H_s}(u)) = 0.$$

This equation comes with the periodicity conditions

$$(2.4) \quad u(s+1, t) = f(u(s, t)).$$

Denote

$$(2.5) \quad f_H := \psi_1^{-1} \circ f \in \text{Symp}(X).$$

(A correct notation would be f_{H_s} , but we stick to f_H for brevity). Suppose the fixed points of f_H are isolated and non-degenerate (that means, for every $x \in \text{Fix } f_H$, $\ker(\text{Id} - df_H(x)) = 0$). Then finite energy solutions to Floer's equation have the following convergence property. There exist points x, y such that

$$(2.6) \quad \lim_{t \rightarrow -\infty} u(s, t) = \psi_s(x), \quad \lim_{t \rightarrow +\infty} u(s, t) = \psi_s(y), \quad x, y \in \text{Fix } f_H.$$

For $x, y \in \text{Fix } f_H$, let $\mathcal{M}(x, y; J_s, H_s)$ be the space of all solutions to Floer's equation (2.3) with limits (2.6). For generic J_s, H_s it is a manifold which is a disjoint union of the k -dimensional pieces $\mathcal{M}^k(x, y; J_s, H_s)$. They can be oriented in a way consistent with gluings; such orientations are called coherent [11]. There is an \mathbb{R} -action on $\mathcal{M}(x, y; J_s, H_s)$. Once a coherent orientation is fixed, $\mathcal{M}^1(x, y; J_s, H_s)/\mathbb{R}$ becomes a collection of points in which every point carries a sign ± 1 attached to it.

The Floer complex associated to $(f; J_s, H_s)$ is the Λ -vector space generated by points in $\text{Fix } f_H$:

$$CF(f; J_s, H_s) := \bigoplus_{x \in \text{Fix } f_H} \Lambda \langle x \rangle.$$

The differential on $CF(f; J_s, H_s)$ is defined on a generator $x \in \text{Fix } f_H$ by:

$$(2.7) \quad \partial(x) = \sum_{\substack{y \in \text{Fix } f_H \\ u \in \mathcal{M}^1(x, y; J_s, H_s)/\mathbb{R}}} \pm q^{\omega(u)} \cdot y.$$

Here the signs are those of the points in $\mathcal{M}^1(x, y; J_s, H_s)/\mathbb{R}$, and

$$(2.8) \quad \omega(u) = \int_{s \in [0, 1]} \int_{t \in \mathbb{R}} u^* \omega \, ds dt.$$

Suppose J_s, H_s and J'_s, H'_s are two regular choices of almost complex structures and Hamiltonians that satisfy the f -periodicity condition (2.1). Choose a family of ω -tame complex structures $J_{s,t}$ and Hamiltonians $H_{s,t}$, $s, t \in \mathbb{R}$ that for each t satisfy condition (2.1) and such that

$$(2.9) \quad J_{s,t} \equiv J'_s, \quad H_{s,t} \equiv H'_s \text{ for } t \text{ near } -\infty, \quad J_{s,t} \equiv J_s, \quad H_{s,t} \equiv H_s \text{ for } t \text{ near } +\infty.$$

We call $J_{s,t}, H_{s,t}$ a homotopy from J'_s, H'_s to J_s, H_s .

Define $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ to be the set of solutions to Floer's continuation equation

$$(2.10) \quad \partial u / \partial t + J_{s,t}(u)(\partial u / \partial s - X_{H_{s,t}}(u)) = 0$$

with periodicity condition (2.4) and asymptotic condition

$$(2.11) \quad \lim_{t \rightarrow -\infty} u(s, t) = \psi_s(x), \quad \lim_{t \rightarrow +\infty} u(s, t) = \psi_s(y), \quad x \in \text{Fix } f_{H'}, \quad y \in \text{Fix } f_H.$$

If $J_{s,t}, H_{s,t}$ are generic, $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ is a manifold. Let $\mathcal{M}^0(x, y; J_{s,t}, H_{s,t})$ be its 0-dimensional component, which is a collection of signed points once coherent orientations (consistent with those for J_s, H_s and J'_s, H'_s) are fixed. Define the continuation map $C_{J_{s,t}, H_{s,t}} : CF(f; J'_s, H'_s) \rightarrow CF(f; J_s, H_s)$ by

$$(2.12) \quad C_{J_{s,t}, H_{s,t}}(x) = \sum_{\substack{y \in \text{Fix } f_H \\ u \in \mathcal{M}^0(x, y; J_{s,t}, H_{s,t})}} \pm q^{\omega(u)} \cdot y.$$

Here $x \in \text{Fix } f_{H'}$. For generic $J_{s,t}, H_{s,t}$, it is a chain map. It induces an isomorphism on homology. So one can actually identify homologies $HF(f; J_s, H_s)$ for all generic J_s, H_s to get one space $HF(f)$. It is called Floer homology of f . It is a $\mathbb{Z}/2$ -graded (see Definition 2.7) vector space over Λ .

2.2. Commuting symplectomorphisms induce action on Floer homology.

As before, X is a compact symplectic manifold satisfying the W^+ condition. Let $f, g : X \rightarrow X$ be two commuting symplectomorphisms. We now define an automorphism $g_{\text{Floer}} : HF(f) \rightarrow HF(f)$.

Pick generic J_s, H_s that satisfy (2.1) to define the complex $CF(f; J_s, H_s)$. Denote

$$(2.13) \quad J'_s := g^* J_s, \quad H'_s := H_s \circ g.$$

We get another complex $CF(f; J'_s, H'_s)$.

Note that $g \circ \psi_1 = \psi'_1$. Let us check that $f_H = f_{H'} \circ g$:

$$f_{H'} \circ g(x) = (\psi'_1)^{-1} f g(x) = (\psi'_1)^{-1} g f(x) = \psi_1^{-1} f(x) = f_H(x)$$

Consequently, g induces a bijection $\text{Fix } f_H \rightarrow \text{Fix } f_{H'}$. Extend it by Λ -linearity to

$$g_{\text{push}} : CF(f; J_s, H_s) \rightarrow CF(f; J'_s, H'_s).$$

Similarly, the composition map $u \mapsto g \circ u$ is an isomorphism

$$\mathcal{M}(x, y; J_s, H_s) \xrightarrow{\cong} \mathcal{M}(g(x), g(y); J'_s, H'_s).$$

This means, if u is a solution to Floer's equation (2.3) with respect to J_s, H_s then $g \circ u$ is a solution of (2.3) with respect to J'_s, H'_s . So g_{push} is tautologically a chain map inducing an isomorphism on homology.

Now fix a homotopy $J_{s,t}, H_{s,t}$ from J'_s, H'_s to J_s, H_s as in (2.9). Consider the composition

$$CF(f; J_s, H_s) \xrightarrow{g_{\text{push}}} CF(f; J'_s, H'_s) \xrightarrow{C_{J_{s,t}, H_{s,t}}} CF(f; J_s, H_s).$$

Definition 2.2 (Action on Floer homology). We denote the composition $C_{J_{s,t}, H_{s,t}} \circ g_{\text{push}}$, as well as the induced automorphism on homology by g_{floer} . In our notation we will frequently suppress the choice of J_s, H_s and write $g_{\text{floer}} : HF(f) \rightarrow HF(f)$.

As a part of this definition, the signs in formula (2.12) for $C_{J_{s,t}, H_{s,t}}$ must come from a coherent orientation as explained in Subsection 2.8 below. In particular, the sign of any element $u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})$, $x \in \text{Fix } f_H$ is canonical and denoted by $\text{sign}(u)$, see Definition 2.15.

Remark 2.3. It is a standard fact that g_{floer} does not depend on the chosen homotopy $J_{s,t}, H_{s,t}$.

Remark 2.4 (An analogue in Morse homology). A similar construction is known in Morse homology [29, 4.2.2]. Suppose $H : X \rightarrow \mathbb{R}$ is a Morse function on a Riemannian manifold (X, g) , and $f : X \rightarrow X$ is a diffeomorphism. Let $C^*(H)$ be the Morse complex of X generated by points in $\text{Crit}(H)$. We define the chain map $f^* : C^*(X) \rightarrow C^*(X)$ as follows.

Pick homotopies H_t from $H \circ f$ to H , g_t from f^*g to g and define $f^* : C^*(H) \rightarrow C^*(H)$ as follows. Take $x, y \in \text{Crit}(H)$ and let the coefficient of $f^*(x)$ by y be the signed count of flowlines of the gradient $\nabla_{g_t} H_t$ going from $f(x)$ to y . The chain map f^* induces an automorphism of $H^*(X)$ known from elementary topology.

Remark 2.5 (Relation to the Seidel invariant). If g is Hamiltonian isotopic to f through symplectomorphisms commuting with f , then one can show $g_{\text{floer}} : HF(f) \rightarrow HF(f)$ is the identity. If g is just Hamiltonian isotopic to f , g_{floer} need not be the identity. It can however be reduced to the Seidel invariant. Take a homotopy g_t , $g_0 = g, g_1 = f$. The path $\gamma_t := g_t^{-1} f g_t$ is actually a loop in $\text{Symp}(X)$: $\gamma(0) = \gamma(1) = f$ because $g^{-1} f g = f$. To this path one associates its Seidel's invariant, $S(\gamma) \in QH(M; \Lambda)$ [33]. Let $*$ be the quantum multiplication $QH(M; \Lambda) \otimes HF(f) \rightarrow HF(f)$. One can check that $g_{\text{floer}}(x) = S(\gamma) * x$ for any $x \in HF(f)$. We will not use this observation, so omit its proof.

2.3. Iterations. If f, g commute then f, g^k also commute for any iteration g^k .

Lemma 2.6. *The following two automorphisms of $HF(f)$ are equal:*

$$(g_{\text{floer}})^k = (g^k)_{\text{floer}}.$$

Proof. We prove it for $k = 2$, the general case is analogous. Take J_s, H_s as in (2.1), J'_s, H'_s pulled by g as in (2.13) and the homotopy $J_{s,t}, H_{s,t}$ as in (2.9). Denote

$$J''_s = g^* J'_s = (g^2)^* J_s, \quad H''_s = H'_s \circ g = H_s \circ g^2.$$

Compare the two compositions given below. The first one induces $(g_{\text{floer}})^2$ on the homological level:

$$\begin{aligned} CF(f; J_s, H_s) &\xrightarrow{g_{\text{push}}} \\ CF(f; J'_s, H'_s) &\xrightarrow{C_{J_s, t, H_s, t}} CF(f; J_s, H_s) \xrightarrow{g_{\text{push}}} CF(f; J'_s, H'_s) \\ &\xrightarrow{C_{J_s, t, H_s, t}} CF(f; J_s, H_s) \end{aligned}$$

The second composition gives $(g^2)_{\text{floer}}$, by a gluing theorem for continuation maps:

$$\begin{aligned} CF(f; J_s, H_s) &\xrightarrow{g_{\text{push}}} \\ CF(f; J'_s, H'_s) &\xrightarrow{g_{\text{push}}} CF(f; J''_s, H''_s) \xrightarrow{C_{J'_s, t, H'_s, t}} CF(f; J'_s, H'_s) \\ &\xrightarrow{C_{J_s, t, H_s, t}} CF(f; J_s, H_s) \end{aligned}$$

By definition of $J'_{s,t}, H'_{s,t}$ (2.13), g maps Floer solutions (2.10) in $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ to those in $\mathcal{M}(g(x), g(y); J'_{s,t}, H'_{s,t})$. This means

$$C_{J'_{s,t}, H'_{s,t}} \circ g_{\text{push}} = g_{\text{push}} \circ C_{J_s, t, H_s, t}.$$

This proves Lemma 2.6. \square

2.4. Supertrace. We continue to use notation from Subsection 2.1.

Definition 2.7. (Grading on Floer's complex) Let $x \in \text{Fix } f_H$. We say $\deg x = 0$ if $\text{sign det}(\text{Id} - df_H(x)) > 0$ and $\deg x = 1$ otherwise.

This makes $CF(f; J_s, H_s)$ a \mathbb{Z}_2 -graded vector space over Λ . Floer's differential has degree 1, so the homology is also \mathbb{Z}_2 -graded: $HF(f) = HF^0(f) \oplus HF^1(f)$.

Definition 2.8 (Supertrace). Let $V = V^0 \oplus V^1$ be a \mathbb{Z}_2 -graded vector space and $\phi : V \rightarrow V$ an automorphism of degree 0, i.e. $\phi(V^0) \subset V^0$, $\phi(V^1) \subset V^1$. Then $STr(\phi) := \text{Tr}(\phi|_{V^0}) - \text{Tr}(\phi|_{V^1})$.

The automorphism g_{floer} from Definition 2.2 has zero degree, so it has a supertrace which is an element of Λ . Supertraces can be computed on the chain level, since all our chain complexes are finite-dimensions, and the following is just a restatement of definitions.

Lemma 2.9. Let X be a symplectic manifold satisfying the W^+ condition and $f, g : X \rightarrow X$ be two commuting symplectomorphisms. Take J'_s, H'_s as in (2.13) and a homotopy $J_{s,t}, H_{s,t}$ from J'_s, H'_s to J_s, H_s as in (2.9). Then

$$STr(g_{\text{floer}} : HF(f) \rightarrow HF(f)) = \sum_{\substack{x \in \text{Fix } f_H, \\ u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})}} (-1)^{\deg x} \cdot \text{sign}(u) \cdot q^{\omega(u)}$$

where $\text{sign}(u) = \pm 1$ is defined in Definition 2.15.

Proof. Pick a generator $x \in \text{Fix } f_H$ of $CF(f; J_s, H_s)$. Rewriting Definition 2.2 of g_{floer} we get

$$g_{\text{floer}}(x) = \sum_{u \in \mathcal{M}^0(g(x), y; J_{s,t}, H_{s,t})} \pm q^{\omega(u)} \cdot y.$$

When we put $x = y$, the sign \pm is substituted by $\text{sign}(u)$ according to Definition 2.2. \square

2.5. Holomorphic sections. It is useful to reformulate the definition of Floer homology using holomorphic sections as in e.g. [37]. If $f : X \rightarrow X$ is a symplectomorphism, denote

$$(2.14) \quad E_f := \frac{X \times \mathbb{R}_{s,t}^2}{(x, s, t) \sim (f(x), s+1, t)}.$$

There is a closed 2-form ω_{E_f} on E_f which comes from $\omega \oplus 0$ on $X \times \mathbb{R}^2$. There is a natural fibration $p : E_f \rightarrow S^1 \times \mathbb{R}$ whose fibers are symplectomorphic to X .

The f -periodicity condition (2.4) on $u : \mathbb{R}^2 \rightarrow X$ means that it can be seen as a section $u : S^1 \times \mathbb{R} \rightarrow E_f$. Floer's equation itself (2.3) is equivalent to u being a holomorphic section with respect to the standard complex structure $j_{S^1 \times \mathbb{R}}$ on $S^1 \times \mathbb{R}$ and an almost complex structure \tilde{J} on E_f . In other words, Floer's equation (2.3) becomes:

$$(2.15) \quad du + \tilde{J} \circ du \circ j_{S^1 \times \mathbb{R}} = 0.$$

The almost complex structure $\tilde{J} := \tilde{J}(J_s, H_s)$ is determined by J_s and H_s , see e.g. [22, Section 8.1]. Analogously, if $J_{t,s}, H_{t,s}$ is a continuation homotopy (2.9), the moduli space $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ consists of sections $u : S^1 \times \mathbb{R} \rightarrow E_f$ that are holomorphic with respect to $j_{S^1 \times \mathbb{R}}$ and an almost complex structure $\tilde{J}(J_{s,t}, H_{s,t})$ on E_f .

2.6. Asymptotic linearised Floer's equation. Let E_f be as in (2.14). We denote by

$$T^v E_f = \ker dp$$

the vertical tangent bundle of E_f . The almost complex structures J_s turn $T^v E_f$ into a complex vector bundle. Take a solution $u(s, t)$ to Floer's equation, $u \in \mathcal{M}(x, y; J_s, H_s)$. We regard it as a section $u(s, t) : S^1 \times \mathbb{R} \rightarrow E_f$ as explained above. The pullback $u^* T^v E_f$ is a complex vector bundle over $S^1 \times \mathbb{R}$. By linearising Floer's equation (2.15), one gets a map

$$(2.16) \quad D_u : H^{1,p}(u^* T^v E_f) \rightarrow L^p(\Omega^{0,1}(u^* T^v E_f)).$$

Here $\Omega^{0,1}(u^* T^v E_f)$ consists of bundle maps $T(S^1 \times \mathbb{R}) \rightarrow u^* T^v E_f$ which are complex-antilinear with respect to \tilde{J} and the standard complex structure on $S^1 \times \mathbb{R}$.

We know from (2.6) that u extends to $S^1 \times \{\pm\infty\}$: $u(s, -\infty) = \psi_s(x)$ where ψ_s is the flow (2.2) of X_{H_s} . (The same is true of $t \rightarrow +\infty$ and the point y . We will now speak of $t \rightarrow -\infty$ only.) Choose a complex trivialisation

$$(2.17) \quad \Phi_x : u^* T^v E_f|_{S^1 \times \{-\infty\}} \rightarrow S^1 \times \mathbb{R}^{2n}.$$

We choose a single trivialisation for each point x ; this is possible because $u(s, -\infty) = \psi_s(x)$. The operator D_u is asymptotic, as $t \rightarrow -\infty$, to the operator

$$(2.18) \quad L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s) : \quad H^{1,p}(S^1 \times \mathbb{R}, \mathbb{R}^{2n}) \rightarrow L^p(S^1 \times \mathbb{R}, \mathbb{R}^{2n}).$$

Here J_0 is the standard complex structure on \mathbb{R}^{2n} , $A(s)$ is a map $S^1 \rightarrow \text{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ taking values in symmetric matrices. It is known that $A(s)$ is determined by J_s, H_s , the point x and the chosen trivialisation $\Phi(x)$. It does not depend on u as long as the $t \rightarrow -\infty$ asymptotic of u stays fixed. A reference for these facts is (among others) the thesis of Schwarz [30, Definition 3.1.6, Theorem 3.1.31]. Although that thesis only considers the case $f = \text{Id}$, all results we use are valid for general f .

Lemma 2.10 ([30, proof of Lemma 3.1.33]). *Consider the operator*

$$J_0 \partial/\partial s + A(s) : \quad C^\infty(S^1, \mathbb{R}^{2n}) \rightarrow C^\infty(S^1, \mathbb{R}^{2n}).$$

There is a family of linear maps $\Psi(s) : [0; 1] \rightarrow \text{Sp}(\mathbb{R}^{2n})$ such that

$$(2.19) \quad (J_0 \partial/\partial s + A(s)) \Psi(s) = 0, \quad \Psi(0) = \text{Id}$$

and $\Psi(1) : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ coincides, under the trivialisation Φ_x (2.17), with the differential $df_H(x)$. \square

Remark 2.11. We identify $S^1 = \mathbb{R}/\mathbb{Z}$ so points of the circle $s = 0$ and $s = 1$ are the same. The statement about $\Psi(1)$ in the lemma above makes sense because $u(0, -\infty) = x$ where $x \in \text{Fix } f_H$, see (2.6) and (2.2). So $df_H(x)$ acts on $T_x X = u^* T^v E_f|_{(0, -\infty)}$. The trivialisation (2.17) identifies this space with \mathbb{R}^{2n} .

Remark 2.12. Given $\Psi(s) : [0; 1] \rightarrow \text{Sp}(\mathbb{R}^{2n})$, by solving (2.19) we get

$$(2.20) \quad A(s) = -J_0(\partial/\partial s \Psi(s)) \Psi(s)^{-1}$$

with symmetric $A(s) : [0; 1] \rightarrow \text{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. Conversely, $A(s)$ defines $\Psi(s)$ by solving (2.19) as an ODE.

2.7. An index problem on the torus. The operator $L_{A(s)}$ (2.18) is Fredholm if and only if $\det(\text{Id} - \Psi(1)) = \det(\text{Id} - df_H(x))$ is nonzero. Now, for later use, consider variables (s, t) belonging to the torus $S^1 \times S^1$ instead of the cylinder $S^1 \times \mathbb{R}$. The same formula (2.18) gives the operator

$$L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s) : C^\infty(S^1 \times S^1, \mathbb{R}^{2n}) \rightarrow C^\infty(S^1 \times S^1, \mathbb{R}^{2n})$$

which is now Fredholm of index 0 for any family of symmetric matrices $A(s) : S^1 \rightarrow \text{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. For the remainder of this subsection, $L_{A(s)}$ denotes the operator on $S^1 \times S^1$ and not on the cylinder.

Lemma 2.13. *Let $A(s) : S^1 \rightarrow \text{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ be a family of symmetric matrices. Suppose $A(s)$ and $\Psi(s)$ satisfy (2.19). Then $\dim \ker L_{A(s)} = \dim \ker(\text{Id} - \Psi(1))$.*

Proof. Any $\xi(s, t) \in \ker L_{A(s)}$ must be independent of t , see [30, Proof of Lemma 3.1.33], so we write $\xi(s, t) \equiv \xi(s)$. The equation on $\xi(s)$ becomes $(J_0 \partial/\partial s + A(s)) \xi(s) = 0$. This is an ODE whose solutions are of form $\xi(s) = \Psi(s)v$ for some $v \in \mathbb{R}^{2n}$ by (2.19). There are no other solutions by the uniqueness theorem for ODEs, as $v \in \mathbb{R}^{2n}$

sweep out all initial conditions. In addition we must have $\xi(1) = \xi(0)$, meaning $\Psi(1)v = v$. \square

Let $A_0(s), A_1(s) : S^1 \rightarrow \text{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ be two families of symmetric matrices with $L_{A_0(s)}, L_{A_1(s)}$ injective. Choose a generic smooth homotopy $A_\tau(s)$ between them, $\tau \in [0; 1]$. Define $\text{sign}(L_{A_0(s)}, L_{A_1(s)}) = (-1)^\epsilon$ where

$$(2.21) \quad \epsilon = \sum_{\tau \in [0; 1]} \dim_{\mathbb{R}} \ker L_{A_\tau(s)}.$$

This sum contains a finite number of nonzero terms as $L_{A_\tau(s)}$ are generically injective. The sum does not depend modulo 2 on the chosen path.

Lemma 2.14. *For $A_0(s), A_1(s)$ as above and $\Psi_0(s), \Psi_1(s)$ satisfying (2.19), we have $\text{sign}(L_{A_0(s)}, L_{A_1(s)}) = \text{sign det}(\text{Id} - \Psi_0(1)) \cdot \text{sign det}(\text{Id} - \Psi_1(1))$.*

Proof. For $i = 0, 1$ denote $\tilde{\Psi}_i(s) = e^{s \log \Psi_i(1)}$ so that $\tilde{\Psi}_i(0) = \Psi_i(0) = \text{Id}$ and $\tilde{\Psi}_i(1) = \Psi_i(1)$. Let us compute $\tilde{A}_i(s)$ from $\tilde{\Psi}_i(s)$ using (2.20): $\tilde{A}_i(s) = -J_0(\partial/\partial s \tilde{\Psi}(s))\tilde{\Psi}(s)^{-1} = -J_0 \log \Psi_i(1)$. We see it is a constant s -independent symmetric matrix $\tilde{A}_i(s) \equiv \tilde{A}_i$. We claim that $\text{sign}(L_{A_i(s)}, L_{\tilde{A}_i}) = +1$. Indeed, choose the homotopy $(\Psi_i)_\tau(s) = e^{\tau s \log \Psi_i(1)} e^{(1-\tau) \log \Psi_i(s)}$ from $\Psi_i(s)$ to $\tilde{\Psi}_i(s)$ with fixed endpoints $\Psi_i(0), \Psi_i(1)$. Passing from $(\Psi_i)_\tau(s)$ to $(A_i)_\tau(s)$ by formula (2.20) we get linear homotopy $(A_i)_\tau(s) = \tau A_i(s) + (1-\tau)\tilde{A}_i$ from $A_i(s)$ to \tilde{A}_i . The corresponding operator $L_{(A_i)_\tau(s)}$ is injective for all τ by Lemma 2.13 because we are given $\ker(\text{Id} - \Psi_i(1)) = 0$.

Let us compute $\text{sign}(L_{\tilde{A}_0}, L_{\tilde{A}_1})$ for two constant matrices $\tilde{A}_i(s) \equiv \tilde{A}_i$, $i = 0, 1$. By linear algebra, one can find a smooth path of matrices \tilde{A}_τ from \tilde{A}_0 to \tilde{A}_1 such that $(-1)^{\sum \dim \ker \tilde{A}_\tau} = \text{sign det } \tilde{A}_0 \cdot \text{sign det } \tilde{A}_1$. It is easy to see that $\ker \tilde{A}_\tau = \ker L_{\tilde{A}_\tau}$, compare proof of Lemma 2.13. Consequently, $\text{sign}(L_{\tilde{A}_0}, L_{\tilde{A}_1}) = \text{sign det } \tilde{A}_0 \cdot \text{sign det } \tilde{A}_1$.

Combining the above,

$$\begin{aligned} \text{sign}(L_{A_0(s)}, L_{A_1(s)}) &= \text{sign}(L_{A_0(s)}, L_{\tilde{A}_0}) \cdot \text{sign}(L_{\tilde{A}_0}, L_{\tilde{A}_1}) \cdot \text{sign}(L_{\tilde{A}_1}, L_{A_1(s)}) = \\ &= \text{sign det } \tilde{A}_0 \cdot \text{sign det } \tilde{A}_1. \end{aligned}$$

Finally, recall $\tilde{A}_i = -J_0 \log \Psi_i(1)$ and observe that $\text{sign det } \log \Psi_i(1) = \text{sign det}(\text{Id} - \Psi_i(1))$. This completes the proof. \square

2.8. Signs for the action on Floer homology. Let f, g be two commuting symplectomorphisms. The goal of this section is to complete Definition 2.2 of the action $g_{\text{floer}} : HF(f) \rightarrow HF(g)$ by specifying the signs appearing there.

Pick regular J_s, H_s to define the moduli space $\mathcal{M}(x, y; J_s, H_s)$ of solutions to Floer's equation (2.3). We get Floer's complex $CF(f; J_s, H_s)$. For each $x \in \text{Fix } f_H$, pick a trivialisation Φ_x (2.17). Then for each x we get a unique asymptotic linearised operator $L_{A_x(s)}$ (2.18).

Let J'_s, H'_s be pulled by g (2.13) and $J_{s,t}, H_{s,t}$ be a homotopy (2.9). Let $u \in \mathcal{M}^0(g(x), y; J_{s,t}, H_{s,t})$ be a solution to Floer's continuation equation, where $x, y \in \text{Fix } f_H$ so that $g(x) \in \text{Fix } f_{H'}$. Consider the linearised Floer's operator D_u . It is very

similar to the operator considered in Subsection 2.6. As $t \rightarrow +\infty$, D_u is asymptotic to $L_{A_y(s)}$ because for t close to $+\infty$, $J_{s,t}, H_{s,t}$ are equal to J_s, H_s . On the other hand, as $t \rightarrow -\infty$, we can write down D_u in the g -induced trivialisation $\Phi_x \circ dg$ of $u^*TE_f|_{u(-\infty, s)}$. We claim that D_u is asymptotic, as $t \rightarrow -\infty$, to $L_{A_x(s)}$. Indeed, the asymptotic operator is determined by the following data: the fixed point $g(x)$, the chosen trivialisation $\Phi_x \circ dg$, and $J_{s,t}, H_{s,t}$ which equal $g^*J_s, H_s \circ g$ for t close to $-\infty$. We see this whole data is pulled by g from the data x, Φ_x, J_s, H_s which defines the asymptotic linearised operator $A_x(s)$. Clearly, pullback by g does not change the linearised operator at all, so $A_x(s)$ is indeed the $t \rightarrow -\infty$ asymptotic to D_u .

The outcome is that the set $\{L_{A_x(s)}\}_{x \in \text{Fix } f_H}$ of asymptotic operators to D_u for $u \in \mathcal{M}(x, y; J_s, H_s)$ (these are solutions to Floer's equations for the differential on $CF(f)$, without the second symplectomorphism g involved) is identical to the set of asymptotic operators to D_u for $u \in \mathcal{M}(g(x), y; J_{s,t}, H_{s,t})$ (these are solutions to Floer's continuation equation).

Consequently, the usual definition of coherent orientations [11] on $\mathcal{M}(x, y; J_s, H_s)$ can be applied without any change to orient $\mathcal{M}(g(x), y; J_{s,t}, H_{s,t})$, $x, y \in \text{Fix } f_H$. In Definition 2.2, we pick such a coherent orientation on $\mathcal{M}(g(x), y; J_{s,t}, H_{s,t})$. Instead of repeating the complete definition of coherent orientations, we only recall a piece relevant to the signs appearing in Lemma 2.9 regarding the supertrace of g_{floer} .

Coherent orientations are not unique, but the sign any coherent orientation associates to a point $u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})$, $x \in \text{Fix } f_H$, is canonical. We explain its definition following [11] and [22, Appendix A]. As we have seen, D_u is asymptotic as $t \rightarrow \pm\infty$ to the same operator

$$L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s).$$

(where $A(s) = A_x(s)$ in notation of the previous paragraphs). Choose a homotopy L_τ from D_u to $L_{A(s)}$, $\tau \in [0; 1]$ such that L_τ are Fredholm operators asymptotic to $L_{A(s)}$ as $t \rightarrow \pm\infty$.

Definition 2.15. For $u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})$, $x \in \text{Fix } f_H$, define $\text{sign}(u) = (-1)^\epsilon$ where

$$\epsilon = \sum_{\tau \in [0; 1]} \dim_{\mathbb{R}} \ker L_\tau.$$

Because operators L_τ have index 0, the sum is well-defined and does not depend modulo 2 on the chosen path. Let us repeat that, as part of Definition 2.2, these signs appear in Lemma 2.9.

2.9. Holomorphic sections over the torus. Subsection 2.5 explained that solutions to Floer's equation are holomorphic sections of a fibration $p : E_f \rightarrow S^1 \times \mathbb{R}$. The monodromy of this fibration around S^1 equals f .

Let f, g be two commuting symplectomorphisms of X . In this subsection we define a fibration $p : E_{f,g}^{1,R} \rightarrow T^{1,R}$ over a 2-torus $T^{1,R}$. The monodromies of this fibration are f and g around the two loops of the torus. After that we recall how to count its holomorphic sections, see [22] for details. This construction is a crucial ingredient for proving Theorem 1.1.

Consider the torus

$$T^{1,R} := \frac{[0; 1] \times [-R; R]}{(s, \{-R\}) \sim (s, \{R\}), (\{0\}, t) \sim (\{1\}, t)}$$

Equip $T^{1,R}$ with the complex structure $j^{1,R}$ coming from the standard one on $[0; 1] \times \sqrt{-1}[-R; R] \subset \mathbb{C}$. Define

$$E_{f,g}^{1,R} := \frac{X \times [0; 1] \times [-R; R]}{(x, s, \{-R\}) \sim (g(x), s, \{R\}), (x, \{0\}, t) \sim (f(x), \{1\}, t)}$$

Here $x \in X$, $s \in [0; 1]$, $t \in [-R; R]$. Because $fg = gf$, there's a fibration $p : E_{f,g}^{1,R} \rightarrow T^{1,R}$ and a fiberwise symplectic closed 2-form $\omega_{E_{f,g}^{1,R}}$ coming from one on X .

Fix a generic almost complex structure \tilde{J} on $E_{f,g}$ such that \tilde{J} is $\omega_{f,g}^{1,R}$ -tame on the fibers and the projection $p : E_{f,g}^{1,R} \rightarrow T^{1,R}$ is $(\tilde{J}, j^{1,R})$ -holomorphic.

Let $\mathcal{M}(j^{1,R}, \tilde{J})$ be the space of all $(j^{1,R}, \tilde{J})$ -holomorphic sections $u : T^{1,R} \rightarrow E_{f,g}^{1,R}$:

$$(2.22) \quad du + \tilde{J}(u) \circ du \circ j^{1,R} = 0.$$

For generic \tilde{J} , it is a smooth manifold that breaks into components of different dimensions. This manifold has a canonical orientation, in particular its 0-dimensional part $\mathcal{M}^0(j^{1,R}, \tilde{J})$ consists of signed points. We will now describe how these signs are defined.

Let $u \in \mathcal{M}^0(j^{1,R}, \tilde{J})$. Consider the linearised equation (2.22) at u ,

$$D_u : C^\infty(u^*T^v E_{f,g}^{1,R}) \rightarrow \Omega^{0,1}(u^*T^v E_{f,g}^{1,R}).$$

Here $T^v E_{f,g}^{1,R} = \ker dp$ and $u^*T^v E_{f,g}^{1,R}$ is a complex bundle over the torus $T^{1,R}$. Because u has index zero, this bundle has Chern number 0 and hence is trivial; fix its trivialisation. Together with the holomorphic coordinates (s, t) on $T^{1,R}$, it induces a trivialisation of $\Omega^{0,1}(u^*T^v E_{f,g}^{1,R}) = \mathbb{R}^{2n}$. In this trivialisation, D_u is a 0-order perturbation of the Cauchy-Riemann operator:

$$(2.23) \quad D_u = \partial/\partial t + J_0 \partial/\partial s + A(s, t) : C^\infty(T^{1,R}, \mathbb{R}^{2n}) \rightarrow C^\infty(T^{1,R}, \mathbb{R}^{2n})$$

where $A(s, t) : T^{1,R} \rightarrow \text{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. This is the same operator as considered in Subsection 2.7, except that now $A(s, t)$ can depend on t as well as on s . The operator D_u is always Fredholm of index 0.

Fix, once and for all, an injective operator of the above form, for example

$$L_{\text{Id}} = \partial/\partial t + J_0 \partial/\partial s + \text{Id}.$$

Find a smooth homotopy of operators L_τ , $\tau \in [0; 1]$, from D_u to L_{Id} , by deforming the 0-order part $A(s, t)$ to Id .

Definition 2.16 (cf. [22, p. 51 and Appendix A]). For $u \in \mathcal{M}^0(j^{1,R}, \tilde{J})$, define $\text{sign}(u) := (-1)^\epsilon$ where

$$\epsilon = \sum_{\tau \in [0; 1]} \dim_{\mathbb{R}} \ker L_\tau.$$

For $u \in \mathcal{M}^0(j^{1,R}, \tilde{J})$, denote $\omega(u) := \int_{T^{1,R}} u^* \omega_{E_{f,g}^{1,R}}$. The following is well-known.

Proposition 2.17.

$$\sharp \mathcal{M}^0(j^{1,R}, \tilde{J}) := \sum_{u \in \mathcal{M}^0(j^{1,R}, \tilde{J})} \text{sign}(u) \cdot q^{\omega(u)}$$

is independent of the complex structure $j^{1,R}$ on the torus and of generic \tilde{J} . \square

2.10. Gluing the fibration over the cylinder to the fibration over the torus.

Given a symplectomorphism $f : X \rightarrow X$, we have constructed a fibration $p : E_f \rightarrow S^1 \times \mathbb{R}$ (2.14). Given two commuting symplectomorphisms $f, g : X \rightarrow X$ and a parameter $R \in \mathbb{R}$, we have constructed a fibration $E_{f,g}^{1,R} \rightarrow T^{1,R}$ over the torus $T^{1,R}$. The fibers of both fibrations are X . Now, there is a map

$$(2.24) \quad E_f \supset p^{-1}(S^1 \times [-R; R]) \rightarrow E_{f,g}^{1,R}$$

It glues the boundary component $p^{-1}(S^1 \times \{R\})$ to the other boundary component $p^{-1}(S^1 \times \{-R\})$ via the symplectomorphism $g : X \rightarrow X$ applied fiberwise along S^1 .

Fix regular J_s, H_s (2.1). As in (2.13), set

$$J'_s = g^* J_s, \quad H'_s = g \circ H_s.$$

Choose a homotopy $J_{s,t}, H_{s,t}$ (2.9) between J'_s, H'_s and J_s, H_s . This homotopy must be t -independent for large and small t ; we assume for convenience

$$J_{s,t} \equiv J'_s, \quad H_{s,t} \equiv H'_s \text{ for } t \leq -R, \quad \text{and} \quad J_{s,t} \equiv J_s, \quad H_{s,t} \equiv H_s \text{ for } t \geq R.$$

Finally, let $\tilde{J} := \tilde{J}(J_{s,t}, H_{s,t})$ be the almost complex structure on E_f from Subsection 2.5, which has the property that solutions to Floer's continuation equation are exactly $(j_{S^1 \times \mathbb{R}}, \tilde{J})$ -holomorphic sections $S^1 \times \mathbb{R} \rightarrow E_f$.

By definition, $\tilde{J}|_{p^{-1}(S^1 \times \{R\})}$ is the g -pullback of $\tilde{J}|_{p^{-1}(S^1 \times \{-R\})}$, which agrees with the gluing (2.24). So \tilde{J} defines a glued almost complex structure $\text{gl}\tilde{J}$ on $E_{f,g}^{1,R}$.

Let us once more recall our notation. $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ consists of holomorphic sections over $S^1 \times \mathbb{R}$ which are solutions to Floer's continuation equation (2.10). $\mathcal{M}(j^{1,R}, \text{gl}\tilde{J}(J_{s,t}, H_{s,t}))$ consists of holomorphic sections over the torus $T^{1,R}$. We come to an important proposition, of which everything but formula (2.25) is well-known.

Proposition 2.18. *For large enough R , there's a bijection called the gluing map and denoted by gl :*

$$\text{gl} : \bigsqcup_{x \in \text{Fix } f_H} \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t}) \xrightarrow{1-1} \mathcal{M}^0(j^{1,R}, \text{gl}\tilde{J}(J_{s,t}, H_{s,t})).$$

It preserves ω -areas:

$$\int_{S^1 \times \mathbb{R}} u^* \omega_{E_f} = \int_{T^{1,R}} \text{gl}(u)^* \omega_{E_{f,g}^{1,R}}$$

and changes the signs from Definitions 2.15, 2.16 by $(-1)^{\deg x}$:

$$(2.25) \quad \text{sign}(u) = \text{sign}(\text{gl}(u)) \cdot (-1)^{\deg x}$$

Here $u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})$, and $\deg x$ is defined in Definition 2.7.

Proof. The existence of the bijection gl is well-known. The map gl is constructed for the case $f = g = \text{Id}$ in [30], see also [5], and that proof carries over to arbitrary f, g . Alternatively, one can adopt general SFT gluing and compactness theorems [6].

Let $u(s, t) \in \mathcal{M}(g(x), x; J_{s,t}, H_{s,t})$. By a smooth homotopy this section can be made t -independent for t close to $-\infty$ and $+\infty$. We can glue it into a smooth section over $T^{1,R}$ by applying (2.24). This gluing preserves ω -areas. The smooth section over $T^{1,R}$ we obtained is smoothly homotopic to $\text{gl}(u)$ and hence has the same ω -area as $\text{gl}(u)$.

Let us explain why gl changes the sign by $(-1)^{\deg x}$. The following informal diagram illustrates the argument below.

$$\begin{array}{ccc}
 \begin{array}{c} \text{over} \\ S^1 \times \mathbb{R} \end{array} & D_u \xrightarrow{\text{sign}(u)} \partial/\partial t + J_0 \partial/\partial s + A(s) & \\
 & \Downarrow \text{gluing} & \\
 \begin{array}{c} \text{over} \\ T^{1,R} \end{array} & D_{\text{gl}(u)} \xrightarrow{\quad} \partial/\partial t + J_0 \partial/\partial s + A(s) \xrightarrow{(-1)^{\deg x}} \partial/\partial t + J_0 \partial/\partial s + \text{Id} & \\
 & \underbrace{\hspace{15em}}_{\text{sign}(\text{gl}(u))} &
 \end{array}$$

Take $u(s, t) \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})$ and consider linearised Floer's operators (2.16) and (2.23):

$$\begin{aligned}
 D_u : \quad & H^{1,p}(S^1 \times \mathbb{R}, \mathbb{R}^{2n}) \rightarrow L^p(S^1 \times \mathbb{R}, \mathbb{R}^{2n}) \\
 D_{\text{gl}(u)} : \quad & C^\infty(T^{1,R}, \mathbb{R}^{2n}) \rightarrow C^\infty(T^{1,R}, \mathbb{R}^{2n}).
 \end{aligned}$$

Take a homotopy L_τ from D_u to the operator (2.18) $L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s)$. By Definition 2.15

$$(2.26) \quad \text{sign}(u) = (-1)^{\sum_\tau \dim \ker L_\tau}$$

Let L_τ^{gl} be a homotopy from $D_{\text{gl}(u)}$ to the analogous operator $L_{A(s)} = \partial/\partial t + J_0 \partial/\partial s + A(s)$ over the torus considered in Subsection 2.7. It is well-known that

$$(2.27) \quad \sum_{\tau \in [0;1]} \dim \ker L_\tau = \sum_{\tau \in [0;1]} \dim \ker L_\tau^{\text{gl}} \pmod{2}.$$

(This is a special case of the fact that orientations of moduli spaces of pseudoholomorphic sections before gluing canonically define orientations on moduli spaces after gluing.)

Now take a homotopy $L_{A_\tau(s)}$ from $L_{A(s)}$ to $L_{\text{Id}} = \partial/\partial t + J_0 \partial/\partial s + \text{Id}$. By Lemma 2.14, Lemma 2.10 and Definition 2.7,

$$(2.28) \quad \sum_{\tau \in [0;1]} \dim \ker L_{A_\tau(s)} = \deg x \pmod{2}.$$

The concatenation of homotopies $L_{A_\tau(s)}$ and L_τ^{gl} is a homotopy from $D_{\text{gl}(u)}$ to $\partial/\partial t + J_0 \partial/\partial s + \text{Id}$. So by Definition 2.16,

$$(2.29) \quad \text{sign}(\text{gl}(u)) = (-1)^{\sum_\tau \dim \ker L_{A_\tau(s)}} \cdot (-1)^{\sum_\tau \dim \ker L_\tau^{\text{gl}}}.$$

Combine (2.26), (2.27), (2.28), (2.29) from this proof to get

$$\text{sign}(u) = \text{sign}(\text{gl}(u)) \cdot (-1)^{\deg x}.$$

This completes the proof. \square

2.11. Proof of the elliptic relation.

Proof of Theorem 1.1. We only need to compile previous statements. By Lemma 2.9 and Proposition 2.18, for sufficiently large R we have

$$STr(g_{\text{floer}}) = \sum_{\substack{x \in \text{Fix } f_H, \\ u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t})}} (-1)^{\deg x} \cdot \text{sign}(u) \cdot q^{\omega(u)} = \sharp \mathcal{M}^0(j^{1,R}, \tilde{J}(J_{s,t}, H_{s,t})).$$

One can repeat all constructions after swapping f and g to get

$$STr(f_{\text{floer}} : HF(g) \rightarrow HF(g)) = \sharp \mathcal{M}^0(j^{R,1}, \tilde{J}_1).$$

Here $j^{R,1} = j^{1, \frac{1}{R}}$ is another complex structure on the torus (which is ‘long’ in the s -direction, while $j^{1,R}$ is ‘long’ in the t -direction), and \tilde{J}_1 some other almost complex structure. Now Theorem 1.1 follows from Proposition 2.17. \square

2.12. Finite order symplectomorphisms. We prove two lemmas about the action of Floer homology when one of the two commuting symplectomorphisms has finite order. After that we prove Proposition 1.4. The proof of the next lemma is an extension of [14, Lemma 7.1].

Lemma 2.19. *Let X be a symplectic manifold satisfying the W^+ condition. Let $g, \phi : X \rightarrow X$ be two commuting symplectomorphisms. Suppose $\phi^k = \text{Id}$ and the fixed point set X^ϕ is a smooth manifold (maybe disconnected, with components of different dimensions). Then*

$$STr(g_{\text{floer}} : HF(\phi) \rightarrow HF(\phi)) = L(g|_{X^\phi}) \cdot q^0 + \sum_i a_i \cdot q^{\omega_i}, \quad \omega_i > 0.$$

In other words, $STr(g_{\text{floer}}) \in \Lambda$ contains only summands with non-negative powers of q , and the q^0 -coefficient is the topological Lefschetz number of $g|_{X^\phi}$. Using the elliptic relation we show later (Corollary 2.21) that the higher order terms $a_i q^{\omega_i}$ actually vanish. This is a separate argument and we first prove the lemma as stated.

Proof. First we construct a Hamiltonian function on X of special form. Let $U(X^\phi)$ be a ϕ -equivariant tubular neighborhood of X^ϕ , $p : U(X^\phi) \rightarrow X^\phi$ the projection and dist a ϕ -invariant function on $U(X^\phi)$ measuring the distance to X^ϕ in some ϕ -invariant metric. Let H_0 be an arbitrary function on X^ϕ . Define

$$H := H_0 \circ p + \text{dist}^2.$$

It is a function on $U(X^\phi)$. Extend this function to X in any way and then average it with respect to ϕ (this will not change the function on $U(X^\phi)$). We denote the result by H again. Note that $H|_{X^\phi} = H_0$ and $\text{Crit}(H_0) = \text{Crit}(H) \cap X^\phi$. For the rest of the proof, H will be a generic function constructed this way; in particular $H|_{X^\phi}$ is generic.

Because ϕ has finite order, we can choose a ϕ -invariant compatible almost complex structure J on X which preserves TX^ϕ , and such that $J|_{X^\phi}$ is arbitrary.

Because J, H are ϕ -invariant, they satisfy (2.1) (with $f = \phi$). Thus Floer's equation (2.3) makes sense for the s -independent data J, H . Denote $J' \equiv g^*J$, $H' \equiv H \circ g$ as in (2.13).

Choose an s -independent homotopy (2.9) H_t, J_t from H' to H (resp. from J' to J). For every t , H_t, J_t must be ϕ -invariant, and as earlier

$$(2.30) \quad H_t = (H_0)_t \circ p + dist^2$$

on $U(X^\phi)$ where $(H_0)_t = (H_t)|_{X^\phi}$ can be arbitrary. Note that in general, it might not be possible to find s -invariant J_t, H_t that would make all solutions of Floer's continuation equation (2.10) regular. However, using [14] we will now argue that some solutions of (2.10) (namely gradient flowlines of H_t) are still generically regular.

Recall that J_t defines the time-dependent metric $\omega(\cdot, J_t \cdot)$ on X by definition of a compatible almost complex structure. If H is a function on X , its gradient and Hamiltonian vector fields are related by: $\nabla H = JX_H$. So s -independent solutions $u(s, t) \equiv x(t)$ of Floer's continuation equation (2.10) are exactly $\omega(\cdot, J_t \cdot)$ -gradient flowlines of H_t :

$$dx(t)/dt - \nabla H_t = 0.$$

The ϕ -periodicity condition (2.4) now reads $\phi(x(t)) = x(t)$ so we are looking only at gradient flowlines inside X^ϕ . Note that every s -independent solution $u(s, t) \equiv x(t)$ of (2.10) has zero area: $\omega(u) = 0$. Recall that solutions of (2.10) are, by notation, elements of $\mathcal{M}(x, y; J_t, H_t)$ where $x \in \text{Fix } \phi_{H'}$ and $y \in \text{Fix } \phi_H$. Also note that $\text{Fix } \phi_H = \text{Crit}(H|_{X^\phi})$, and similarly $\text{Fix } \phi_{H'} = \text{Crit}(H'|_{X^\phi})$.

The following two facts are proved in [14] when H_t, J_t are t -independent and $\phi = \text{Id}$ (that paper is interested in the equations for Floer's differential rather than continuation maps). The proofs are valid in the general case. For example, one can track that the periodicity condition (2.1), which is the only place where ϕ explicitly appears, is not used in the proof of the facts below.

- (1) For any J_t, H_t as above, an s -independent solution $u(s, t) \equiv x(t)$ of (2.10) is regular, i.e. D_u (2.16) is onto, if and only if the $\omega(\cdot, J_t \cdot)$ -gradient flow of H_t is Morse-Smale near X^ϕ [28, Corollary 4.3, Theorem 7.3], compare [14, proof of Theorem 6.1].
- (2) There is $\epsilon > 0$ such that every solution $u(s, t)$ of (2.10) with $\omega(u) < \epsilon$, see (2.8), is s -independent [14, Lemma 7.1].

We claim that the gradient flow of a generic H_t as above is Morse-Smale near X^ϕ . Indeed, we can choose $H_t|_{X^\phi}$ freely, so we can make the flow of $H_t|_{X^\phi}$ to be Morse-Smale. Because H_t is quadratic in the normal direction to X^ϕ (2.30), the stable manifolds of H_t are, near X^ϕ , normal disk bundles over those of $H_t|_{X^\phi}$, and the unstable manifolds of H_t lie in X^ϕ and coincide with those of $H_t|_{X^\phi}$. Consequently, H_t is Morse-Smale near X^ϕ if and only if $H_t|_{X^\phi}$ is Morse-Smale.

By Remark 2.4 or [29, 4.2.2],

$$(2.31) \quad \sum_{\substack{x \in \text{Fix } \phi_H, \\ u \in \mathcal{M}^0(g(x), x; J_s, H_s): \omega(u) \leq 0}} (-1)^{\deg x} \cdot \text{sign}(u) \cdot q^{\omega(u)} = L(g|_{X^\phi}) \cdot q^0.$$

Although the left hand side looks exactly like the expression for $STr(g_{\text{floer}})$ from Lemma 2.9, J_t, H_t are not regular for all continuation equation solutions, while g_{floer} must be computed using regular ones. So we slightly perturb J, H and J_t, H_t by allowing them to depend on s : we get J_s, H_s and $J_{s,t}, H_{s,t}$. We can achieve that all solutions to (2.10) with respect to $J_{s,t}, H_{s,t}$ become regular.

Because s -independent solutions in $\mathcal{M}(x, y; J_t, H_t)$ were already regular, they are in 1-1 correspondence (via the continuation map) with some solutions in $\mathcal{M}(x, y; J_{s,t}, H_{s,t})$ of zero ω -area. By item (2) above, every $u \in \mathcal{M}(x, y; J_{s,t}, H_{s,t})$ with $\omega(u) < \epsilon$ actually has zero area and corresponds to an s -independent solution in $\mathcal{M}(x, y; J_t, H_t)$. (See [14, proof of Proposition 7.4] for this argument.) In view of (2.31) this means

$$\sum_{\substack{x \in \text{Fix } \phi_H, \\ u \in \mathcal{M}^0(g(x), x; J_{s,t}, H_{s,t}): \omega(u) \leq 0}} (-1)^{\deg x} \cdot \text{sign}(u) \cdot q^{\omega(u)} = L(g|_{X^\phi}) \cdot q^0.$$

Lemma 2.19 follows from this equality and Lemma 2.9. \square

Lemma 2.20. *Let X be a symplectic manifold satisfying the W^+ condition. Let $g, \phi : X \rightarrow X$ be two commuting symplectomorphisms. Suppose $\phi^k = \text{Id}$. Then*

$$STr(\phi_{\text{floer}} : HF(g) \rightarrow HF(g)) = a \cdot q^0, \quad \text{where } a \in \mathbb{C} \text{ and } |a| \leq \dim_\Lambda HF(\phi).$$

Proof. By Lemma 2.6 $(\phi_{\text{floer}})^k = \text{Id}$, so all eigenvalues of ϕ_{floer} are among the roots of unity $\sqrt[k]{1} \cdot q^0 \in \Lambda$. The signed sum of these eigenvalues gives $STr(\phi_{\text{floer}})$, and Lemma 2.20 follows. \square

The elliptic relation (Theorem 1.1) and Lemma 2.20 imply the following corollary.

Corollary 2.21. *The terms $a_i \cdot q^{\omega_i}$, $\omega_i > 0$ from Lemma 2.19 actually vanish.* \square

Proof of Proposition 1.4. The proposition follows from Lemma 2.19, Lemma 2.20 and Theorem 1.1. \square

Remark 2.22. As promised in Remark 1.7 we sketch an alternative proof of Proposition 1.4 without appealing to Theorem 1.1. Suppose for simplicity a symplectomorphism $f : X \rightarrow X$ commutes with a symplectic involution ι , f has non-degenerate isolated fixed points, and $d\iota$ acts by $-\text{Id}$ on the normal bundle to its fixed locus X^ι . Choose the zero Hamiltonian perturbation for $HF(f)$ and an almost complex structure which is ι -invariant at points $x \in \text{Fix } f \cap X^\iota$. Then ι_{floer} only counts constant solutions $u(s, t) \equiv x \in \text{Fix } f \cap X^\iota$. (It is clear that the only zero-area solutions are constant, and because $\iota_{\text{floer}}^2 = \text{Id}$ all positive area solutions cancel.) However, the sign associated to a constant solution u is not always positive. The reason is that we must write the linearised Floer's operator D_u in a trivialisation of $u^*T_x X = S^1 \times \mathbb{R} \times T_x X$ which differs by $d\iota(x)$ over the two ends of the cylinder, according to the definition in Subsection 2.8. Consider the splitting $T_x X = T_x X^\iota \oplus N_x X^\iota$ into the $+1$ and -1

eigenspaces of $du(x)$. We can choose the constant trivialisation of $u^*T_x X^\iota$ and get the \mathbb{R} -independent operator on this subspace which by definition carries the positive sign. However, the trivialisation of $u^*N_x X^\iota$ cannot be constant (it can for example be a rotation from Id to $-\text{Id}$ with parameter t), so D_u will not be the canonical \mathbb{R} -invariant operator on $N_x X$ and can carry a nontrivial sign from Definition 2.15. We claim that this sign equals $\text{sign det}(\text{Id} - df(x)|_{N_x X^\iota})$. It can be checked using arguments from Subsection 2.7; a related Lagrangian version of this statement is [36, Lemma 14.11]. Once the signs are known, it is easy to see that $STr(\iota_{\text{floer}}) = L(f|_{\text{Fix } \iota}) \cdot q^0$:

$$\begin{aligned} STr(\iota_{\text{floer}}) &= \sum_{x \in \text{Fix } f \cap X^\iota} (-1)^{\deg x} \cdot \text{sign det}(\text{Id} - df(x)|_{N_x X^\iota}) \cdot q^0 = \\ &\quad \sum_{x \in \text{Fix } f \cap X^\iota} \text{sign det}(\text{Id} - df(x)|_{T_x X^\iota}) \cdot q^0 = L(f|_{\text{Fix } \iota}) \cdot q^0. \end{aligned}$$

The bound $\dim HF(f) \geq L(f|_{\text{Fix } \iota})$ follows as in Lemma 2.20.

2.13. Lagrangian elliptic relation. In this section, we briefly state and prove Theorem 1.9 and Proposition 1.10. Let X be a monotone symplectic manifold, i.e. $[\omega(X)] = \lambda c_1(X)$ as elements of $H^2(X; \mathbb{R})$, $\lambda > 0$. Let $\phi : X \rightarrow X$ be a symplectomorphism, $L_i \subset X$ two connected monotone Lagrangian submanifolds such that $\phi(L_i) = L_i$.

In order to define the action $\phi_{\text{floer}} : HF(L_1, L_2) \rightarrow HF(L_1, L_2)$ we must fix additional data which we now describe, unless the base field has characteristic 2 in which case no additional data is necessary. First, L_i must be oriented, but ϕ need not preserve the orientations. (In Appendix A we use the orientation-reversing case.) Second, L_i must be equipped with spin structures S_i together with isomorphisms $\phi^* S_i \rightarrow S_i$ if $\phi|_{L_i}$ preserves orientation, and $\phi^* S_i \rightarrow \bar{S}_i$ if $\phi|_{L_i}$ reverses orientation, where \bar{S}_i is the following spin structure on \bar{L}_i (that is, on L_i with the opposite orientation). The original spin structure S_i is a trivialisation of TL_i over the 1-skeleton of L_i which extends over the 2-skeleton and agrees with the orientation on L_i . By definition, \bar{S}_i is the composition of the trivialisation S_i with a fixed orientation-reversing isomorphism $\mathbb{R}^n \rightarrow \mathbb{R}^n$, for example the one which multiplies the first coordinate by -1 . We note the desired isomorphisms $\phi^* S_i \rightarrow S_i$ or $\phi^* S_i \rightarrow \bar{S}_i$ always exist if L_i are simply-connected. In [36, Section 14], similar data (defined only for an involution ϕ , with an extra condition on the ‘squares’ of the above isomorphisms, but also allowing non-orientable Lagrangians) was called an equivariant Pin structure.

Pick some J_s, H_s defining Floer homology $HF(L_1, L_2; J_s, H_s, S_i)$, see [24, 12] for a definition in the monotone setting. We have included the choice of spin structures in our notation. The action ϕ_{floer} is the composition $HF(L_1, L_2; J_s, H_s, S_i) \rightarrow HF(L_1, L_2; \phi^* J_s, H_s \circ \phi, \phi^* S_i) \rightarrow HF(L_1, L_2; J_s, H_s, S_i)$. Here the first map is the tautological chain-level map that takes all chain generators and Floer’s solutions to their ϕ -image; we are using that $\phi L_i = L_i$. The second one is the continuation map. We skip the proof of the next lemma.

Lemma 2.23 (cf. [36, Sections (14a) and (14e)]). *If $\phi^k = \text{Id}$ then $(\phi_{\text{floer}})^k = \pm \text{Id}$.* \square

Note that we do not necessarily get $(\phi_{\text{floer}})^k = \text{Id}$ as opposed to Lemma 2.6 and [36, top of p. 310], but having $(\phi_{\text{floer}})^k = \pm \text{Id}$ is enough for future applications.

Choose J_s, H_s (2.1) to define Floer's complex $CF(\phi; J_s, H_s)$. Take the fibration $p : E_\phi \rightarrow S^1 \times [0; +\infty)$ with monodromy ϕ around the circle as in (2.14), but now over the semi-infinite cylinder $S^1 \times [0; +\infty)$ instead of $S^1 \times \mathbb{R}$. It contains the ‘boundary condition’ manifold $S^1 \times L \subset p^{-1}(S^1 \times \{0\})$. The symplectic form on X defines a fiberwise symplectic form ω_{E_ϕ} on E_ϕ . Choose a tame almost complex structure \tilde{J} on E_ϕ which, over $S^1 \times [1; +\infty)$, equals $\tilde{J}(J_s, H_s)$ for some J_s, H_s (see Subsection 2.5), in particular is independent of $t \in [1; +\infty)$.

Take $x \in \text{Fix } \phi_H$ (2.5), that is, a generator of $CF(\phi; J_s, H_s)$. We define $\mathcal{M}^0(L, x)$ to be the set of all zero index \tilde{J} -holomorphic sections $u(s, t) : S^1 \times [0; +\infty) \rightarrow E_\phi$ which are asymptotic, as $t \rightarrow +\infty$, to the Hamiltonian trajectory $\psi_s(x)$ (2.2), and satisfy the Lagrangian boundary condition $u(s, 0) \in S^1 \times L$. Then we define

$$[L]^\phi = \sum_{x \in \text{Fix } \phi_H} \sum_{u \in \mathcal{M}^0(L, x)} \pm q^{\omega(u)} \cdot [x] \in HF(\phi).$$

Here $[x] \in HF(\phi)$ is the homology class of the chain generator x , and $\omega(u) = \int_{S^1 \times [0; +\infty)} u^* \omega_{E_\phi}$. The signs are defined using the chosen spin structures on L_i and coherent orientations for ϕ . This is a version of the open-closed string map, cf. [26].

Next we review the quantum product $HF(\phi) \otimes HF(\phi^{-1}) \rightarrow HF(\text{Id}) \cong QH(X)$. It counts holomorphic sections of a symplectic fibration over S^2 with three punctures and monodromies $\phi, \phi^{-1}, \text{Id}$ around them. The first two punctures serve as inputs from $HF(\phi), HF(\phi^{-1})$, and the third puncture is the output, see [22] for details. If one caps the output puncture by a disk, the count of sections over the resulting twice-punctured sphere (see the lower part of Figure 3(a)), gives the composition $HF(\phi) \otimes HF(\phi^{-1}) \rightarrow HF(\text{Id}) \xrightarrow{\chi} \Lambda$ of the product and the projection χ onto $HF_0(\text{Id}) \cong \Lambda$.

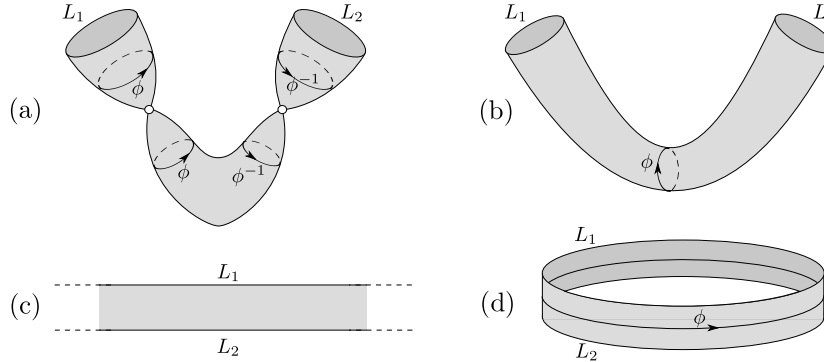


FIGURE 3. Proving the Lagrangian elliptic relation.

Combining the definitions, $\chi([L_1]^\phi * [L_2]^{\phi^{-1}})$ counts holomorphic sections over two cylinders and a twice-punctured sphere which have the same asymptotics over the punctures in two pairs, see Figure 3(a). Here the cylinder $S^1 \times [0; +\infty)$ is seen as a once punctured disk. This count equals the number of sections of a glued fibration

over an annulus, with monodromy ϕ around the core circle, and Lagrangian conditions $S^1 \times L_1, S^1 \times L_2$ over the boundary of the annulus. The annulus carries a fixed ‘long’ complex structure, see Figure 3(b).

On the other hand, $STr(\phi_{\text{floer}})$ counts sections of a trivial fibration over the strip $[0; 1] \times \mathbb{R}$ with Lagrangian boundary conditions $\mathbb{R} \times L_i$ and asymptotics differing by ϕ over $t \rightarrow \pm\infty$, see Figure 3(c). We can glue the fibration over the strip twisting it by ϕ to get a fibration over the annulus which we have already encountered: it carries Lagrangian conditions $S^1 \times L_i$ over the boundary and has monodromy ϕ around the core circle, see Figure 3(d). By gluing, $STr(\phi_{\text{floer}})$ equals to the count of holomorphic sections of this fibration, with a fixed (‘long’, but in the other direction than before) complex structure on the annulus. As the count of sections does not depend on the complex structure on the annulus, we get Theorem 1.9. We omit the discussion of signs which was carried out in detail for the case of commuting symplectomorphisms. The present case can be studied by almost the same arguments if we superficially deform the Lagrangians so that $T_p L_1 = T_p L_2$ for all intersection points $p \in L_1 \cap L_2$, keeping these points isolated, and then pick nondegenerate Hamiltonians H_1, H_2 to compute $HF(L_1, L_2)$.

Let us now explain Proposition 1.10. The most important step is to prove a Lagrangian analogue of Lemma 2.19: if ϕ is a map of finite order with smooth fixed locus X^ϕ and smooth orientable Lagrangian fixed loci $L_i^\phi \subset X^\phi$ then

$$(2.32) \quad \chi([L_1]^\phi * [L_1]^\phi) = ([L_1]^\phi \cdot [L_2]^\phi) \cdot q^0 + \sum_i a_i \cdot q^{\omega_i}, \quad \omega_i > 0.$$

Recall that $[L_1]^\phi \cdot [L_2]^\phi \in \mathbb{Z}$ is the homological intersection of the fixed loci L_1^ϕ, L_2^ϕ inside X^ϕ . (Note that L_i^ϕ are automatically isotropic but not necessarily Lagrangian, although we will only use the case when they are Lagrangian. One can get examples of $(\phi_{\text{floer}})^k = -\text{Id}$ in Lemma 2.23 when dimensions of L_1^ϕ, L_2^ϕ are different.)

In order to count sections of the configuration on Figure 3(a), we must specify the data $J_{s,t}, H_{s,t}$ over our configuration consisting of two half-cylinders $S^1 \times [0; +\infty)$ and a twice-punctured sphere which we will now see as the cylinder $S^1 \times \mathbb{R}$. Similarly to Lemma 2.19, we choose the data to be of special form, namely independent of the basepoint: $J_{s,t} \equiv J, H_{s,t} \equiv H$ (this forces J, H to be ϕ -equivariant). With this data, s -independent ($s \in S^1$) sections become gradient flowlines of the Morse function H inside the fixed locus X^ϕ . Rigid sections over $S^1 \times \mathbb{R}$ are constant, while rigid sections over $S^1 \times [0; +\infty)$ are flowlines from L_i to a critical point of H . In the end, the count of s -independent rigid configurations on Figure 3(a) is $\sum_{x \in \text{Crit}^n(H|_{X^\phi})} ([L_1]^\phi \cdot [\text{Stab}(x)]) ([L_2]^\phi \cdot [\text{Stab}(x)])$ where Crit^n are index n critical points, $n = \frac{1}{2} \dim_{\mathbb{R}} X^\phi$, and Stab are stable manifolds in X^ϕ . This sum equals the intersection $[L_1]^\phi \cdot [L_2]^\phi$.

Finally, one must argue that these configurations of flowlines are regular, and are the only zero area solutions. (There could be other positive area solutions which are not necessarily regular). This is a variation on the lemmas cited in the proof of Lemma 2.19. Then one makes the data J, H regular by allowing them to depend on

s, t and argues that the count of zero area solutions (which were already regular) is preserved.

On the other hand, if ϕ is of finite order then $\phi_{\text{floer}} : HF(L_1, L_2) \rightarrow HF(L_1, L_2)$ is of finite order by Lemma 2.23, and the eigenvalues of ϕ_{floer} are among $\sqrt[k]{1} \cdot q^0$. Consequently, $STr(\phi_{\text{floer}}) = a \cdot q^0$, $|a| \leq \dim_{\Lambda} HF(L_1, L_2)$. Now Theorem 1.9 and formula (2.32) imply Proposition 1.10.

3. VANISHING SPHERES AND DEHN TWISTS

Let Y be a Kähler manifold with a Kähler form ω . Take a very ample holomorphic line bundle $\mathcal{L} \rightarrow Y$. Let $X \subset Y$ be a smooth divisor in the linear system $|\mathcal{L}|$.

In this section we define $|\mathcal{L}|$ -vanishing Lagrangian spheres in the symplectic manifold $(X, \omega|_X)$. They exist if the line bundle $\mathcal{L} \rightarrow X$ has zero defect (see below) and are then unique up to symplectomorphism.

Throughout this section, we denote by $D \subset \mathbb{C}$ the unit complex disk.

3.1. Lefschetz fibrations and vanishing cycles. This subsection reviews well-known material, see e.g. [36].

Definition 3.1 (Lefschetz fibration with a unique singularity). Suppose E is a smooth manifold, Ω a closed 2-form on E and $\pi : E \rightarrow D$ is a smooth proper map. The triple (E, Ω, π) is called a Lefschetz fibration with a unique singularity if there is a point $p \in E$ (without loss of generality, we assume $\pi(p) = 0 \in D$), and a neighbourhood $U(p)$ such that:

- π is regular outside $U(p)$, and the restriction of Ω on the regular fibers of π is symplectic;
- there exists a complex structure on $U(p)$ with a holomorphic chart x_1, \dots, x_n such that

$$\pi(x_1, \dots, x_n) = x_1^2 + \dots + x_n^2;$$

- $\Omega|_{U(p)}$ is Kähler with respect to the above complex structure.

All fibers $E_t := \pi^{-1}(t)$ contain a Lagrangian sphere, uniquely defined up to Hamiltonian isotopy. Let us sketch its construction. The fibers E_t are symplectomorphic to each other by parallel transport with respect to the Ω -induced connection on E . It suffices to construct a Lagrangian sphere in E_t for small $t \in \mathbb{R}_+$. Define $L \subset U(p) \subset E$ by the equation

$$x_1^2 + \dots + x_n^2 = t, \quad x_i \in \mathbb{R}$$

Clearly $L \subset E_t$ and is a Lagrangian sphere for $t \in \mathbb{R}_+$ with respect to the standard symplectic structure Ω_0 on $U(p) \subset \mathbb{C}^n$. However, our Kähler form $\Omega|_{U(p)}$ is not necessarily standard. Following [35, Lemma 1.6], write $\Omega|_{U(p)} = \Omega_0 + dd^c f$ and deform f to 0 in a smaller neighborhood of p . This deforms Ω to Ω_0 in a neighborhood of p via forms that are Kähler on $U(p)$. The fiber E_t will stay symplectic during this deformation. By Moser's lemma, the standard Lagrangian sphere L can be mapped to a Lagrangian sphere in $(E_t, \Omega|_{E_t})$. (We will use this argument once more in the proof of Lemma 4.8.) By symplectic parallel transport, the sphere L can be transported between different fibers E_t .

Definition 3.2 (vanishing Lagrangian sphere). Let $E \rightarrow D$ be a Lefschetz fibration with a unique singularity. A Lagrangian sphere in a smooth fiber E_t is called vanishing for the fibration $E \rightarrow D$ if it is Hamiltonian isotopic to the one constructed above.

3.2. Defect of a line bundle.

Definition 3.3 (Defect of a line bundle). Let Y be a complex manifold and $\mathcal{L} \rightarrow Y$ a very ample holomorphic line bundle, giving an embedding $Y \subset (\mathbb{P}^N)^*$ where $\mathbb{P}^N = \mathbb{P}H^0(Y, \mathcal{L})$. The discriminant variety $\Delta \subset \mathbb{P}^N$ is the dual variety to Y . It parameterises all hyperplanes in $(\mathbb{P}^N)^*$ which are tangent to $Y \subset \mathbb{P}^N$. Equivalently, it parameterises all singular divisors in the linear system $\mathbb{P}H^0(Y, \mathcal{L})$. The defect of \mathcal{L} is the number

$$\text{def } \mathcal{L} = N - 1 - \dim \Delta \geq 0.$$

Line bundles usually have zero defect; for us it is useful to note the following.

Lemma 3.4 ([3, page 532]). *Suppose $\mathcal{L} \rightarrow Y$ is a very ample line bundle. If $\text{def } \mathcal{L} \geq 1$, there exists a smooth rational curve $l \subset Y$ such that $\mathcal{L} \cdot l = 1$.* \square

For completeness, let us sketch a proof. Recall that points in Δ^{reg} correspond to generic hyperplanes $H \subset (\mathbb{P}^N)^*$ which are not transverse to Y . If $\text{def } \mathcal{L} \geq 1$, for such a hyperplane $H \in \Delta^{reg}$ the contact locus $(H \cap Y)^{sing}$ is a linear $\mathbb{P}^{\text{def } \mathcal{L}}$ [41, Theorem 1.18]. Take any line $l \cong \mathbb{P}^1$ in H . Obviously it intersects a generic smooth hyperplane section $\tilde{H} \cap Y$ transversely at a single point, which means $\mathcal{L} \cdot l = 1$.

Corollary 3.5. *Suppose $\mathcal{L} \rightarrow Y$ is a very ample line bundle. For any $d \geq 2$, $\text{def } \mathcal{L}^{\otimes d} = 0$.* \square

3.3. $|\mathcal{L}|$ -vanishing spheres in divisors. Recall $D \subset \mathbb{C}$ is the unit disk.

Definition 3.6 (Total space of a family of divisors). Let Y be a Kähler manifold and $\mathcal{L} \rightarrow Y$ a very ample line bundle.

Take a holomorphic embedding $u : D \rightarrow \mathbb{P}H^0(Y, \mathcal{L}) = |\mathcal{L}|$. Then every point $t \in D$ defines a divisor $X_{u(t)} \subset Y$. We call $\{X_{u(t)}\}_{t \in D}$ a family of divisors. The total space of the family $\{X_{u(t)}\}_{t \in D}$ is

$$E := \{(x, u(t)) : x \in X_t, t \in D\} \subset Y \times \mathbb{P}H^0(Y, \mathcal{L}).$$

The restriction of the product Kähler form from $Y \times \mathbb{P}H^0(Y, \mathcal{L})$ to E makes E a Kähler manifold. There is a canonical projection $\pi : E \rightarrow D$ whose fibers are $X_{u(t)}$.

In future we shall write $\{X_t\}_{t \in D}$ instead of $\{X_{u(t)}\}_{t \in D}$.

Definition 3.7 ($|\mathcal{L}|$ -vanishing Lagrangian sphere in a divisor). Let Y be a Kähler manifold and $\mathcal{L} \rightarrow Y$ a very ample line bundle with zero defect, and with $\dim \mathbb{P}H^0(Y, \mathcal{L}) \geq 2$.

Let $\Delta \subset \mathbb{P}H^0(Y, \mathcal{L})$ be the discriminant variety from Definition 3.3. Let $u : D \rightarrow \mathbb{P}H^0(Y, \mathcal{L})$ be a holomorphic embedding such that $u(0) \in \Delta^{reg}$, $u(t) \notin \Delta$ for $t \neq 0$, and the intersection of $u(D)$ with Δ^{reg} is transverse. Let $\pi : E \rightarrow D$ be as in Definition 3.6.

By [21, 1.8], $\pi : E \rightarrow D$ is a Lefschetz fibration with a unique singular point over $t = 0$ (in particular, X_0 has a single node). The vanishing sphere $L \subset X_1$ of this fibration is called an $|\mathcal{L}|$ -vanishing sphere.

Obviously, every smooth divisor in the linear system $|\mathcal{L}|$ contains an $|\mathcal{L}|$ -vanishing sphere (if \mathcal{L} has zero defect). Two different maps $u, u' : D \rightarrow H^0(Y, \mathcal{L})$ with $u(1) = u'(1)$ can give two $|\mathcal{L}|$ -vanishing spheres in X_1 which are not Hamiltonian isotopic and even not homologous to each other, compare Lemma 4.1. However, $|\mathcal{L}|$ -vanishing spheres are unique up to symplectomorphism.

Lemma 3.8. *Let $\mathcal{L} \rightarrow Y$ be a very ample line bundle over a Kähler manifold Y , $\text{def } \mathcal{L} = 0$. Suppose X, X' are two smooth divisors in the linear system $|\mathcal{L}|$ and $L \subset X$, $L' \subset X'$ are two $|\mathcal{L}|$ -vanishing Lagrangian spheres. Then there is a symplectomorphism $\psi : X \rightarrow X'$ such that $\psi(L) = L'$.*

This lemma is probably well-known, but we don't have a clear reference for it so we prove it here. An auxiliary lemma is required.

Lemma 3.9. *Let $\pi : X \rightarrow D \times [0; 1]$ be a smooth map and Ω a closed 2-form on X . Suppose that for every $s \in [0; 1]$, $X_{D;s} := \pi^{-1}(D \times \{s\})$, equipped with the restriction of Ω , is a Lefschetz fibration over D with a unique singularity over $0 \in D$. (In particular, fibers of π are symplectic.) For $t \in D$, $s \in [0; 1]$ denote by $X_{t;s}$ the fiber $\pi^{-1}(\{t\} \times \{s\})$. Let $L_0 \subset X_{1;0}$ (resp. $L_1 \subset X_{1;1}$) be the vanishing cycle of the Lefschetz fibration $X_{D;0}$ (resp. $X_{D;1}$). Then there is a symplectomorphism $\psi : X_{1;0} \rightarrow X_{1;1}$ such that $\psi(L_0) = L_1$.*

Proof. One can choose a smooth family of spheres $L_s \subset X_{1;s}$ such that L_s is vanishing for the fibration on $X_{D;s}$, and L_0, L_1 are the given spheres. This is easily seen from our definition or from [36, proof of Lemma 16.2].

Next, let $\phi_s : X_{1;0} \rightarrow X_{1;s}$ be the parallel transport with respect to Ω [36, Section 15a] along the s -direction. This parallel transport is well-defined because $X_{1;s}$ are smooth and symplectic.

Now look at $\phi_s(L_0)$ and L_s . These are two Lagrangian spheres in $X_{1;s}$. We observed above that L_s depends smoothly on s ; obviously, so does $\phi_s(L_0)$. For $s = 0$, the two spheres coincide with L_0 , so they remain C^∞ close to each other for sufficiently small but positive value $s = s'$, and consequently they are Hamiltonian isotopic inside $X_{1;s'}$. By composing $\phi_{s'}$ with this Hamiltonian isotopy, we get $\psi_{s'} : X_{1;0} \rightarrow X_{1;s'}$ taking L_0 to $L_{s'}$. Continuing this way, in a finite number of steps we will get the desired symplectomorphism ψ between $X_{1;0}$ and $X_{1;1}$ taking L_0 to L_1 . \square

Proof of Lemma 3.8. Let $u, u' : D \rightarrow \mathbb{P}H^0(Y, \mathcal{L})$ be two holomorphic maps as in Definition 3.7, and denote $X = X_{u(1)}$, $X' = X_{u'(1)}$.

By Definition 3.7, $u(0), u'(0) \in \Delta^{reg}$. Since Δ^{reg} is connected, one can find a path $\alpha(s) \in \Delta^{reg}$ from $u(0)$ to $u'(0)$, $s \in [0; 1]$. Next one can find an s -parametric family of holomorphic disks $u_s : D \rightarrow \mathbb{P}H^0(Y, \mathcal{L})$ such that $u_0 = u$, $u_1 = u'$, $u_s(0) \in \Delta^{reg}$ and $u_s(D)$ intersects Δ^{reg} transversely. Consider the space

$$E := \{(x, u_s(t)) : t \in D, s \in [0; 1], x \in X_{u_s(t)}\} \subset Y \times \mathbb{P}H^0(Y, \mathcal{L}).$$

It carries a closed 2-form which is the restriction of the product of the Kähler forms on Y and $\mathbb{P}H^0(Y, \mathcal{L})$. There is also a canonical projection $E \rightarrow D \times [0; 1]$. With this symplectic form and projection, E satisfies conditions of Lemma 3.9. This lemma

provides the desired symplectomorphism $\psi : X \rightarrow X'$ taking an given $|\mathcal{L}|$ -vanishing sphere in X to a given one in X' . \square

3.4. Dehn twists. We recall the definition of Dehn twists from [36, Section (16c)]. First, one defines the Dehn twist as a compactly supported symplectomorphism of T^*S^n . Fix the standard round metric on S^n . Let $|\xi|$ be the norm function on T^*S^n . It is non-smooth at the 0-section. Away from the 0-section, its Hamiltonian flow is the normalised geodesic flow. Take a function $b(r) : \mathbb{R} \rightarrow \mathbb{R}$ with compact support and such that $b(r) - b(-r) = -r$. The Dehn twist $\tau : T^*S^n \rightarrow T^*S^n$ is the 2π -flow of the Hamiltonian function $b(|\xi|)$. It extends smoothly to the 0-section by the antipodal map, thanks to the special form of $b(r)$. As result, τ is a compactly supported symplectomorphism of T^*S^n . We will not require the following theorem (and Corollary 3.13 below), but it is worth stating because it is relevant in view of Theorem 1.2.

Theorem 3.10. (1) τ has infinite order in $\text{Symp}^c(T^*S^n)/\text{Ham}^c(T^*S^n)$, the group of compactly-supported symplectomorphisms of T^*S^n modulo compactly-supported symplectic isotopy.
 (2) If n is even, τ has finite order in $\pi_0\text{Diff}^c(T^*S^n)$, the group of compactly-supported diffeomorphisms of T^*S^n modulo compactly-supported isotopy [19]. \square

When $n = 2$ it is further known that τ generates $\pi_0\text{Symp}^c(T^*S^2) \cong \mathbb{Z}$, and τ^2 is smoothly isotopic to Id in $\text{Diff}^c(T^*S^2)$ [37], see also [2, Theorem 1.21].

Next, if $L \subset X$ is a Lagrangian submanifold in any symplectic manifold, a neighborhood of L in X is symplectomorphic to a neighborhood of the 0-section in T^*S^n . So one can pull back τ using this symplectomorphism and then extend it by the identity to get a map $\tau_L : X \rightarrow X$. It is a symplectomorphism uniquely defined up to Hamiltonian isotopy (once a parameterisation of L is fixed), supported in a neighborhood of L .

Definition 3.11 (Dehn twist). The symplectomorphism $\tau_L : X \rightarrow X$ is called the Dehn twist around L .

Lemma 3.12 (Picard-Lefschetz formula, [21]). If $\dim_{\mathbb{R}} X = 2n$ and $L \subset X$ is a Lagrangian sphere, then $(\tau_L)_*$ acts by Id on $H_i(X)$, $i \neq n$. For any $[A] \in H_n(X)$,

$$(\tau_L)_*[A] = [A] - \epsilon([L] \cdot [A])[L].$$

Here $\epsilon = (-1)^{\frac{1}{2}n(n-1)}$. Consequently:

- (1) if n is even, $(\tau_L)_*^2$ acts by Id on $H_*(X)$.
- (2) if n is odd and $[L] \in H_n(X; \mathbb{R})$ is nonzero, then $(\tau_L)_*$ is an automorphism of infinite order of $H_*(X)$. \square

Summarising Theorem 3.10(2) and Lemma 3.12(2), we arrive to the following well-known statement.

Corollary 3.13. Let $\dim X_{\mathbb{R}} = 2n$ be a compact symplectic manifold and $L \subset X$ a Lagrangian sphere nonzero in $H_n(X; \mathbb{R})$.

- (1) If n is even, τ_L has finite order in $\pi_0\text{Diff}(X)$,

(2) if n is odd, τ_L has infinite order in $\pi_0 \text{Diff}(X)$. \square

The next lemma relates Dehn twists and Lefschetz fibrations, see [36, (15b)] for details.

Lemma 3.14 ([35, 36]). *Let (E, Ω, π) be a Lefschetz fibration with a unique singularity. Let E_1 be its regular fiber and $L \subset E_1$ a vanishing Lagrangian sphere. Then the Dehn twist $\tau_L : E_1 \rightarrow E_1$ is Hamiltonian isotopic to the symplectic monodromy map $E_1 \rightarrow E_1$ obtained from the symplectic parallel transport applied to the fibers E_t along the circle $t \in \partial D$.* \square

Remark 3.15. Let X be a symplectic manifold and $L \subset X$ be a Lagrangian sphere; assume L is nonzero in $H_n(X)$. There are three main previously known cases when τ_L has infinite order in $\text{Symp}(X)/\text{Ham}(X)$ (if X is noncompact, consider $\text{Symp}^c(X)/\text{Ham}^c(X)$ instead):

- (1) $\frac{1}{2} \dim_{\mathbb{R}} X$ is odd, as explained above;
- (2) X is exact with contact type boundary, and L is exact (Seidel, unpublished);
- (3) X is Calabi-Yau, and there is another Lagrangian sphere L' intersecting L once transversely [34].

Let $X = \text{Bl}_k \mathbb{P}^2$ be the blowup of \mathbb{P}^2 in k generic points, $2 \leq k \leq 8$, with the monotone symplectic form, and $L \subset X$ be any Lagrangian sphere. Seidel [37] showed that τ_L has order 2 in $\text{Symp}(X)/\text{Ham}(X)$ when $k = 2, 3, 4$ and has order greater than 2 when $k = 5, 6, 7, 8$, but did not prove it was infinite. Note that $X = \text{Bl}_6 \mathbb{P}^2$ is the cubic surface $X \subset \mathbb{P}^3$, to which Theorem 1.2 applies.

4. CONSTRUCTING INVARIANT LAGRANGIAN SPHERES

In this section we prove Proposition 4.2 below. It will later be used to prove Theorem 1.12 and Proposition A.10.

The following lemma is essentially known. It can be used to prove the simple case of Theorem 1.2 when $\dim_{\mathbb{C}} X$ is odd.

Lemma 4.1. *Let \mathcal{L} be a very ample line bundle over a Kähler manifold Y . For any $d \geq 3$, every smooth divisor $X \subset Y$ in the linear system $|\mathcal{L}^{\otimes d}|$ contains two $|\mathcal{L}^{\otimes d}|$ -vanishing Lagrangian spheres L_1, L_2 that intersect once and transversely.*

The following technical proposition should be considered as an equivariant version of Lemma 4.1. It will be used to prove the harder case of Theorem 1.12 when $\dim_{\mathbb{C}} X$ is even.

Proposition 4.2. *Let \mathcal{L} be a very ample line bundle over a Kähler manifold Y , and let $\iota : Y \rightarrow Y$ be a holomorphic involution which lifts to an automorphism of \mathcal{L} . Suppose the fixed locus Y^{ι} is smooth (maybe disconnected with components of different dimensions), and ι is non-degenerate (i.e. acts by $-\text{Id}$ on the normal bundle to Y^{ι}). Fix $d \geq 3$.*

Pick a connected component $\tilde{\Sigma}$ of $Y^{\iota} \subset Y$, $\dim \tilde{\Sigma} \geq 2$.

Let $H^0(Y, \mathcal{L}^{\otimes d})_{\pm}$ denote the ± 1 -eigenspace of the involution on $H^0(Y, \mathcal{L}^{\otimes d})$ induced by ι . Let Π_{\pm} be as in Theorem 1.12. Suppose one of the following:

(a) d is even;

(b) d is odd,

$\tilde{\Sigma} \subset \Pi_+$, and

there is a smooth divisor in the linear system $\mathbb{P}H^0(Y, \mathcal{L}^{\otimes d})_+$.

(Alternatively, suppose Case (b) holds with symbols $+$ replaced by $-$.)

Then there is a smooth divisor X in the linear system $|\mathcal{L}^{\otimes d}|$ and two $|\mathcal{L}^{\otimes d}|$ -vanishing Lagrangian spheres $L_1, L_2 \subset X$ such that:

(1) $\iota(X) = X$, $\Sigma := X \cap \tilde{\Sigma}$ is smooth, $\dim \Sigma = \dim \tilde{\Sigma} - 1$

(2) $\iota(L_1) = L_1$, $\iota(L_2) = L_2$;

(3) L_1, L_2 intersect transversely at a single point which belongs to Σ ;

(4) $L_i^t = L_i \cap \Sigma$ are Lagrangian spheres in Σ , $i = 1, 2$.

(5) for $i = 1, 2$ one can choose a symplectomorphism τ_{L_i} of X representing the Hamiltonian isotopy class of the Dehn twist around L_i such that τ_{L_i} commutes with ι , and $\tau_{L_i}|_{X^t}$ is the Dehn twist around L_i^t .

4.1. A_2 chains of Lagrangian spheres from A_2 fibrations.

Definition 4.3 (A_2 chain of Lagrangian spheres). Let X be a symplectic manifold. A pair (L_1, L_2) of two Lagrangian spheres in X is called an A_2 -chain if L_1 and L_2 intersect at a single point, and the intersection is transverse.

We have discussed earlier that one can construct Lagrangian spheres (vanishing cycles) from Lefschetz fibrations. Similarly, one can get A_2 chains of Lagrangian spheres from fibrations with slightly more complicated singularities.

Definition 4.4 (A_2 fibration). Denote by $D \subset \mathbb{C}$ the open unit disk, and by $B_\epsilon \subset \mathbb{C}$ the open disk of radius ϵ . Both disks are centered at 0.

Suppose E is a smooth manifold, Ω a closed 2-form on E and $\pi : E \rightarrow D$ is a smooth map. The triple (E, Ω, π) is called an A_2 fibration if there is a point $p \in E$ (without loss of generality, we assume $\pi(p) = 0 \in D$), and a neighbourhood $U(p)$ such that:

- all but a finite number of fibers of π are regular, and the restriction of Ω is symplectic on them;
- there exists a complex structure on $U(p)$ with a holomorphic chart x_1, \dots, x_n , $x_i \in B_\epsilon$ such that

$$\pi(x_1, \dots, x_n) = x_1^2 + \dots + x_{n-1}^2 + h(x_n)$$

where $h(x_n)$ is holomorphic;

- $h(x_n)$ has at least 3 roots within $B_{\epsilon/2}$, counted with multiplicities;
- for any $x_n \in B_{\epsilon/2}$, $\sqrt{h(x_n)} \in B_{\epsilon/2}$;
- $\Omega|_{U(p)}$ is Kähler with respect to the above complex structure.

Remark 4.5. The definition allows π to have singularities outside of $U(p)$. They will not play any role. Also, the definition does not require $p : E \rightarrow D$ to be a proper map, so the fibers E_t need not be symplectomorphic as we may not be able to integrate the parallel transport vector fields. The generality of this definition is

slightly unusual, but it makes no difference to the local construction of A_2 chains of Lagrangian spheres in Lemma 4.8.

In order to prove Proposition 4.2, we need to introduce A_2 fibrations with involutions.

Definition 4.6 (Involutive A_2 fibration). Let (E, Ω, π) be an A_2 fibration. It is called an involutive A_2 fibration with involution $\iota : E \rightarrow E$ if ι is non-degenerate (meaning $\ker(\text{Id} - d\iota) = T_x E^\iota$ for each $x \in E^\iota$), $\iota^* \Omega = \Omega$, $\pi \iota = \pi$, and there's in the holomorphic chart from Definition 4.4 we have in addition:

$$\iota(x_1, \dots, x_l, x_{l+1}, \dots, x_n) = (-x_1, \dots, -x_l, x_{l+1}, \dots, x_n)$$

for some $l < n$. We denote by E^ι the fixed locus of ι .

Remark 4.7. It follows from this definition that $\pi|_{E^\iota} : E^\iota \rightarrow D$ is also an A_2 fibration. Note that $x \in E^\iota$ is regular for π if and only if it is regular for $\pi|_{E^\iota}$. Indeed, as ι is non-degenerate we can decompose $T_x E = T_x E^\iota \oplus N_x$ where N_x is the (-1) -eigenspace of $d\iota(x)$. Since $\pi \iota = \pi$, $N_x \subset \ker d\pi(x)$. So $\text{rk } d\pi(x) = \text{rk } d\pi(x)|_{T_x E^\iota}$.

Consequently, for a regular fiber E_t , its fixed locus E_t^ι is smooth.

The following is a slight refinement of [18, Lemma 6.12].

Lemma 4.8. *Let $\pi : E \rightarrow D$ be an A_2 fibration. Then for every sufficiently small $t \in D$ such that the fiber $E_t := \pi^{-1}(t)$ is smooth, E_t contains an A_2 chain of Lagrangian spheres.* \square

We will use the following equivariant analogue of this lemma.

Lemma 4.9. *Let $\pi : E \rightarrow D$ be an involutive A_2 fibration with an involution ι . Then for every sufficiently small $t \in D$ such that the fiber $E_t := \pi^{-1}(t)$ is smooth, E_t contains an A_2 chain of Lagrangian spheres (L_1, L_2) which satisfy properties (2)–(5) from Proposition 4.2 where we put $X := E_t$ and Σ to be the connected component of E_t^ι which belongs to the connected component of the point p in E^ι . We repeat these properties for convenience:*

- (2) $\iota(L_1) = L_1$, $\iota(L_2) = L_2$,
- (3) L_1, L_2 intersect transversely at a single point which belongs to Σ ,
- (4) $L_i^\iota = L_i \cap \Sigma$ are Lagrangian spheres in Σ , $i = 1, 2$,
- (5) for $i = 1, 2$ one can choose $\tau_{L_i} : X \rightarrow X$ such that τ_{L_i} commutes with ι , and $\tau_{L_i}|_{X^\iota} = \tau_{L_i^\iota}$,

Remark 4.10. $\dim \Sigma = l - 1$ for l as in Definition 4.6.

Proof of Lemma 4.8. Let $U'(p) \subset U(p)$ be the ball around p given by $|x_i| < \epsilon/2$, $i = 1, \dots, n$. We observe that it suffices to assume $\Omega|_{U'(p)}$ to be the standard symplectic form Ω_0 on \mathbb{C}^n in the holomorphic chart (x_1, \dots, x_n) from Definition 4.4. It is generally not possible to make $\Omega|_{U(p)}$ standard by a holomorphic change of coordinates preserving π . But we can follow the argument of [35, Lemma 1.6]: there is a function f on $U(p)$ such that $\Omega|_{U(p)} = \Omega_0 + dd^c f$. We can deform f to 0 on $U'(p)$ while leaving f unchanged outside of $U(p)$. Let f_r be such a homotopy and define $\Omega_r := \Omega$ outside of $U(p)$, and $\Omega_r|_{U(p)} := \Omega_0 + dd^c f_r$. Observe that $\Omega_0 = \Omega$

and $\Omega_1|_{U'(p)}$ is standard in the holomorphic chart (x_1, \dots, x_n) from Definition 4.4. Suppose we have proved the lemma for (E, Ω_1, π) . Then we claim the lemma also holds for (E, Ω_0, π) . Note that for any r the fibers $(E_t, \Omega_r|_{E_t})$ are symplectic and the cohomology class of $\Omega_r|_{E_t}$ is constant, so the fibers are actually symplectomorphic to each other for any r . Once we have found an A_2 chain of spheres in E_t which are Lagrangian with respect to $\Omega_1|_{E_t}$, we can apply the symplectomorphism to get an A_2 chain of Lagrangian spheres in $(E_t, \Omega_0|_{E_t})$.

Consequently, from now on we may assume $\Omega|_{U'(p)}$ is standard. The condition that E_t is smooth means the equation $h(x_n) = t$ has no multiple roots with $x_n \in B_{\epsilon/2}$. By Definition 4.4 the equation $h(x_n) = 0$ has at least 3 roots with $x_n \in B_{\epsilon/2}$. So for sufficiently small t the equation $h(x_n) = t$ also has at least 3 distinct roots with $x_n \in B_{\epsilon/2}$. Choose three distinct roots, say $z_1, z_2, z_3 \in B_{\epsilon/2}$: $h(z_i) = t$. Let $\gamma_{12} \subset B_{\epsilon/2}$ be a path from z_1 to z_2 whose interior avoids roots of $h - t$. Define

$$L_1 := \bigsqcup_{t \in \gamma_{12}} \{(x_1, \dots, x_n) \in B_{\epsilon/2} \cap \pi^{-1}(t) : |x_i| \in \mathbb{R} \cdot \sqrt{-h(t)}\}.$$

This is a smooth Lagrangian sphere with respect to the standard symplectic form on \mathbb{C}^n . Similarly, let $\gamma_{23} \subset B_{\epsilon/2} \subset \mathbb{C}$ be a path from z_2 to z_3 and define L_2 by the same formula replacing γ_{12} by γ_{23} . If γ_{12} and γ_{23} are transverse at their common endpoint z_2 , then (L_1, L_2) is an A_2 chain of Lagrangian spheres by [18, Lemma 6.12]. Note that L_1, L_2 lie in $U'(p)$ by the fourth condition in Definition 4.4. \square

Proof of Lemma 4.9. We use notation from the proof of Lemma 4.8. Arguing as in that proof ι -invariantly, we can again assume Ω is standard on $U'(p)$. The formulas for L_1, L_2 are invariant under the change $x_i \mapsto -x_i$, $i \leq l$, so L_1, L_2 are ι -invariant. This proves property (2) from Proposition 4.2. Next, we already know L_1 intersects L_2 transversely at a single point. This point has coordinates $x_1 = 0, \dots, x_{n-1} = 0$, $x_n = z_2$. (Recall z_2 is a root of $h(x_n) - t$). This point is ι -invariant. Moreover, it obviously belongs to the connected component of point p in E^ι , so property (3) from Proposition 4.2 holds. Property (4) is true because E^ι locally around π is given by $x_1 = \dots = x_l = 0$ and so $L_i \cap \Sigma$ are transverse Lagrangians by the same reason as L_i are. By their local construction, the L_i do not intersect the connected components of E_t^ι other than Σ .

It remains to explain property (5). Let $S^{n-1} \subset \mathbb{R}^n$ be the standard unit sphere. Let ι_0 be the involution on S^n which changes the sign of the first k coordinates on \mathbb{R}^n . It naturally extends to an involution ι_0 on T^*S^n . It is not hard to check there is an (ι, ι_0) -equivariant diffeomorphism $V(L_1) \rightarrow V(S^n)$ where $V(L_1)$ is an ι -invariant tubular neighbourhood of $L_1 \subset X$ and $V(S^n)$ is an ι_0 -invariant tubular neighborhood of the zero-section in T^*S^n . Then there is also an (ι, ι_0) -equivariant symplectomorphism $V(L_1) \rightarrow V(S^n)$, by an equivariant analogue of the Weinstein tubular neighborhood theorem. The Dehn twist in T^*S^n is ι_0 -equivariant by definition. Its pullback via the equivariant symplectomorphism $V(L_1) \rightarrow V(S^n)$ is the desired ι -equivariant Dehn twist inside E_t . \square

4.2. A_2 fibrations of divisors from projective embeddings. One way of getting an A_2 fibration is to embed all of its fibers E_t as divisors $E_t = X_t \subset Y$ into a single

Kähler manifold Y . This idea can be used to prove Lemma 4.1. We will now do this ι -invariantly to prove Proposition 4.2 (necessary for Theorem 1.2) with the help of Lemma 4.8.

Proof of Proposition 4.2. Let us recall the setting. We have a very ample bundle $\mathcal{L} \rightarrow Y$ over a Kähler manifold Y . We are given a holomorphic involution $\iota : Y \rightarrow Y$ which lifts to an involution on \mathcal{L} . This means ι induces a linear involution on $H^0(Y, \mathcal{L})^*$ splitting it into the direct sum of ± 1 eigenspaces denoted by $H^0(Y, \mathcal{L})_{\pm}^*$. The projectivisations of these eigenspaces are denoted by $\Pi_{\pm} \subset \mathbb{P}H^0(Y, \mathcal{L})^*$. We also denote $\mathbb{P}^N := \mathbb{P}H^0(Y, \mathcal{L})^*$, and the ι -induced involution on \mathbb{P}^N by $\iota_{\mathbb{P}^N}$. The fixed locus of $\iota_{\mathbb{P}^N}$ is $\Pi_+ \sqcup \Pi_- \subset \mathbb{P}^N$.

Because \mathcal{L} is very ample, we have an embedding $Y \subset \mathbb{P}^N$, $\mathcal{L} = \mathcal{O}_Y(1) := \mathcal{O}_{\mathbb{P}^N}(1)|_Y$, Y is invariant under $\iota_{\mathbb{P}^N}$ and $\iota_{\mathbb{P}^N}|_Y = \iota$, and

$$Y^{\iota} = (Y \cap \Pi_+) \sqcup (Y \cap \Pi_-).$$

Let $\tilde{\Sigma}$ be the given connected component of Y^{ι} (smooth by assumption), denote $\dim \tilde{\Sigma} = l$. Then $\tilde{\Sigma} \subset \Pi_{\epsilon}$ where ϵ is one of the two symbols: $+$ or $-$. We will also denote by ϵ the correspondingly signed number ± 1 .

Choose homogeneous coordinates $(x_0 : \dots : x_l : x_{l+1} : \dots : x_N)$ on \mathbb{P}^N with the following properties:

- (1) $\iota_{\mathbb{P}^N}(x_0 : \dots : x_l : x_{l+1} : \dots : x_N) = (\epsilon x_0 : \dots : \epsilon x_l : \pm x_{l+1} : \dots : \pm x_N)$;
- (2) $(1 : 0 : \dots : 0) \in \tilde{\Sigma}$
- (3) the plane spanned by (x_0, \dots, x_l) (other coordinates are set to 0) is the tangent plane to $\tilde{\Sigma}$ at $(1 : 0 : \dots : 0)$;
- (4) for some $n \geq l$, the plane spanned by (x_0, \dots, x_n) (other coordinates are set to 0) is the tangent plane to Y at $(1 : 0 : \dots : 0)$.

The third property implies that x_0, \dots, x_l , seen as sections in $H^0(\mathcal{O}_{\mathbb{P}^N}(1))$, belong to the ϵ -eigenspace of ι . This is in agreement with the first property. So coordinates with the above properties exist.

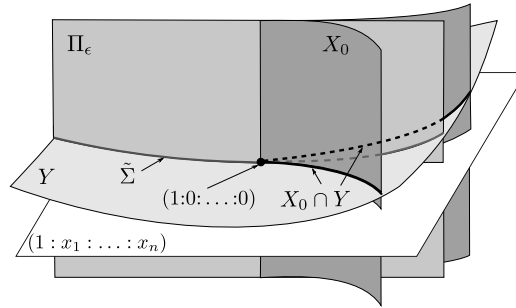


FIGURE 4. A divisor X_0 from the family X_t constructed in the proof of Proposition 4.2.

In the affine chart $x_0 = 1$, coordinates (x_1, \dots, x_n) are local coordinates for Y near the origin. In the chart $x_0 = 1$, write (see Figure 4):

$$X_t := x_1^3 + x_2^2 + \dots + x_n^2 - t.$$

We want X_t to be a section of $\mathcal{O}_{\mathbb{P}^N}(d)$, so in projective coordinates we set

$$X_t := x_0^{d-3}x_1^3 + x_0^{d-2}(x_2^2 + \dots + x_n^2) - tx_0^d.$$

From property (1) of the coordinates x_i , we see that $X_t \circ \iota = \epsilon^d X_t$ as polynomials. In other words:

- (a) if d is even, $X_t \in H^0(\mathcal{O}_{\mathbb{P}^N}(d))_+$;
- (b) if d is odd, $X_t \in H^0(\mathcal{O}_{\mathbb{P}^N}(d))_\epsilon$.

For all t , the divisors $\{X_t = 0\}$ and $\{X_t = 0\} \cap Y$ are reducible and hence singular. We want to smooth the family $\{X_t = 0\} \cap Y$ so that a generic divisor in this t -family becomes nonsingular.

Suppose d is even. Then the linear system $H^0(\mathcal{O}_{\mathbb{P}^N}(d))_+$ has no base locus as it contains all monomials x_i^d . Then $H^0(\mathcal{O}_Y(d))_+ = H^0(Y, \mathcal{L}^{\otimes d})_+$ has no base locus too. By Bertini's theorem in characteristic 0, there exists $F \in H^0(\mathcal{O}_{\mathbb{P}^n}(d))_+$ such that the divisor $\{F = 0\} \cap Y$ is smooth.

Suppose d is odd. Then the linear systems $H^0(\mathcal{O}_{\mathbb{P}^N}(d))_\pm$ have nonempty base loci, namely Π_\mp . Therefore it is not apriori clear that these linear systems contain a smooth divisor. This condition is included in the assumptions of Proposition 4.2, Case (b). Let $F \in H^0(\mathcal{O}_{\mathbb{P}^n}(d))_\epsilon$ be a polynomial such that $\{F = 0\} \cap Y$ is smooth.

The rest of the proof is the same for even and odd d . For all generic $\delta \in \mathbb{C}$, the divisors $\{X_t + \delta F = 0\} \cap Y$ are smooth except for a finite number of t 's.

Recall that (x_1, \dots, x_n) is a holomorphic chart for Y around $(1 : 0 : \dots : 0)$. There's another chart $\tilde{x}_1, \dots, \tilde{x}_n$ in which $\{X_t + \delta F\} \cap Y$ are given by:

$$h(\tilde{x}_1) + \tilde{x}_2^2 + \dots + \tilde{x}_n^2 - t + c = 0$$

where $h(\tilde{x}_1)$ is close to \tilde{x}_1^3 (when δ is small) and c is a small constant. Moreover, the change of coordinates from x_i to \tilde{x}_i is ι -equivariant. This follows from an equivariant version of the holomorphic Morse splitting lemma [1].

Consider the family $\{X_t + \delta F = 0\} \cap Y$ of divisors in Y , $t \in D$. They are ι -invariant and belong to the linear system $|\mathcal{L}^{\otimes d}|$. Let $E \rightarrow D$ be the total space of this family, Definition 3.6. It could be singular, in this case throw away its singular locus to get E_0 . The involution ι turns $E_0 \rightarrow D$ into an involutive fibration in the sense of Definition 4.6. So by Lemma 4.9, a smooth divisor in the family $\{X_t + \delta F = 0\} \cap Y$ has a pair of Lagrangian spheres (L_1, L_2) that satisfy properties (2)–(5) of Proposition 4.2. It is easy to see that Lemma 4.9 constructs L_1, L_2 which are $|\mathcal{L}^{\otimes d}|$ -vanishing.

Let's check property (1). We have to show that smooth divisors $\{X_t + \delta F = 0\} \cap Y$ intersect $\Sigma = \tilde{\Sigma} \cap Y$ transversely. Suppose $X := \{X_t + \delta F = 0\} \cap Y$ intersects Σ non-transversely at one point p , so $T_p \Sigma \subset T_p X$ (the tangent spaces are taken inside Y). Then $T_p X$ contains $\dim \Sigma$ positive (+1) eigenvalues of $d\iota$. Then the same holds for all intersection points $X \cap \Sigma$, and hence $T_p \Sigma \subset T_p X$ for any $p \in X \cap \Sigma$. But in a neighborhood of $(1 : 0 : \dots : 0)$ the intersection $X \cap \Sigma$ is transverse. This is easily verified in the local chart (x_1, \dots, x_n) from above. So X intersects Σ transversely everywhere. Similarly, every other connected component of Y^ι either intersects X transversely or is contained in X . \square

5. PROOFS OF THEOREMS ABOUT LAGRANGIAN SPHERES IN DIVISORS

Proof of Theorem 1.12. Apply Proposition 4.2 to $Y, \mathcal{L}, \tilde{\Sigma}$ given in the hypothesis of Theorem 1.12. Proposition 4.2 returns an $|\mathcal{L}^{\otimes d}|$ -divisor $X \subset Y$ and $|\mathcal{L}^{\otimes d}|$ -vanishing Lagrangian spheres $L_1, L_2 \subset X$ as described there. Because $|\mathcal{L}^{\otimes d}|$ -vanishing spheres are unique up to symplectomorphism (Lemma 3.8), it suffices to show that τ_{L_1} has infinite order in $\text{Symp}(X)/\text{Ham}(X)$ for these given X and L_1 . To show this, we compute the Lefschetz number of $\tau_{L_1}^{2k} \tau_{L_2}^{2k}|_{X^\iota} = \tau_{L_1}^{2k} \tau_{L_2}^{2k}$ on $H^*(X^\iota)$, where X^ι is the fixed locus of the involution ι on X . Recall that $\Sigma = \tilde{\Sigma} \cap X$ is a connected component of X^ι . We are given that $\dim \tilde{\Sigma}$ is even, so $\dim \Sigma = \dim \tilde{\Sigma} - 1$ is odd. Denote $X^\iota = \Sigma \sqcup \Sigma_0$ where Σ_0 is all other connected components. We identify $H^*(X^\iota)$ with $H_*(X^\iota)$ via Poincaré duality.

Consider homology classes $[L_1^\iota], [L_2^\iota] \in H_*(\Sigma)$, $[L_1^\iota] \cdot [L_2^\iota] = 1$. Using the Picard-Lefschetz formula (see Subsection 3.4) and property (5) from Proposition 4.2, we write down the actions of Dehn twists on the 2-dimensional vector space $\text{span}\{[L_1^\iota], [L_2^\iota]\} \subset H_*(X^\iota)$. Denote $s = \dim_{\mathbb{C}} \Sigma$ and $\epsilon = (-1)^{\frac{1}{2}s(s-1)}$.

$$(\tau_{L_1^\iota})_*^{2k} : \begin{pmatrix} 1 & k(1+(-1)^{s-1})\epsilon \\ 0 & 1 \end{pmatrix}, \quad (\tau_{L_2^\iota})_*^{2k} : \begin{pmatrix} 1 & 0 \\ k(1+(-1)^{s-1})\epsilon & 1 \end{pmatrix},$$

Now use that $s = \dim_{\mathbb{C}} \Sigma$ is odd and get

$$\text{Str} \left((\tau_{L_1^\iota})_*^{2k} (\tau_{L_2^\iota})_*^{2k} |_{\text{span}\{[L_1^\iota], [L_2^\iota]\}} \right) = -4k^2 - 2.$$

(The negative signs appear because we are computing the supertrace). If s were even, we would get the constant 2 instead.

We can extend $[L_1^\iota], [L_2^\iota]$ to a basis of $H_*(X^\iota)$ in which all other elements have zero intersection with $[L_1^\iota], [L_2^\iota]$. By the Picard-Lefschetz formula, $(\tau_{L_i^\iota})_*$ act by Id on the rest of the basis. Consequently, the Lefschetz number

$$L \left((\tau_{L_1^\iota})_*^{2k} (\tau_{L_2^\iota})_*^{2k} \right) = -4k^2 + c$$

where c is a constant independent of k . By Proposition 1.4,

$$\dim_{\Lambda} HF(\tau_{L_1}^{2k} \tau_{L_2}^{2k}) \geq |-4k^2 + c|.$$

Suppose $\tau_{L_1}^{2k}$ is Hamiltonian isotopic to Id for some $k > 0$. Then $\tau_{L_2}^{2k}$ is also Hamiltonian isotopic to Id because by Lemma 3.8 there is a symplectomorphism of X taking L_1 to L_2 . Then the product $\tau_{L_1}^{2k} \tau_{L_2}^{2k}$ is also Hamiltonian isotopic to Id. This also holds if we substitute k by its multiple, and this contradicts to the growth of Floer homology above. Consequently $\tau_{L_1}^{2k}$ cannot be Hamiltonian isotopic to Id when $k \neq 0$, meaning τ_{L_1} has infinite order in the group $\text{Symp}(X)/\text{Ham}(X)$. \square

Next we prove Lemma 1.13. It follows from a strong Bertini theorem which we now quote.

Theorem 5.1 ([9, Corollary 2.4]). *Let Y be a compact smooth complex manifold and S an effective linear system of divisors on Y . Let B be the base locus of S . If B is reduced and nonsingular, and $\dim B < \frac{1}{2} \dim Y$, then a general divisor in S is smooth.* \square

If B is disconnected, the dimensional inequality must hold for every connected component of B .

Proof of Lemma 1.13. We repeat the beginning of proof of Proposition 4.2. We have $Y \subset \mathbb{P}^N$ and $\mathcal{L}^{\otimes d} = \mathcal{O}_{\mathbb{P}^N}(d)|_Y$. The involution ι acts on sections of \mathcal{L} and so acts on \mathbb{P}^N by a linear involution $\iota_{\mathbb{P}^N}$, and $Y \subset \mathbb{P}^N$ is invariant under it. Find homogeneous coordinates $(x_0 : \dots : x_N)$ such that

$$\iota_{\mathbb{P}^N}(x_0 : \dots : x_l : x_{l+1} : \dots : x_N) = (x_0 : \dots : x_l : -x_{l+1} : \dots : -x_N).$$

Recall that d is odd. Then $H^0(\mathcal{O}_{\mathbb{P}^N}(d))_+$ consists of degree- d polynomials which are sums of monomials of the following form:

$$x_0^{\text{odd}} \dots x_l^{\text{odd}} x_{l+1}^{\text{even}} \dots x_N^{\text{even}}.$$

Here *even* or *odd* denote the parity of a power. The base locus of the linear system $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^N}(d))_+$ is given by

$$x_0 = 0, \dots, x_l = 0$$

and so coincides with Π_- . The base locus B of $\mathbb{P}H^0(Y, \mathcal{L}^{\otimes d})_+$ is therefore $\Pi_- \cap Y$. It is smooth because we are given that Y^ι is smooth. We are also given that $\dim B < \frac{1}{2} \dim Y$. Finally, we know that ι is non-degenerate and $\iota_{\mathbb{P}^N}|_{\Pi_-} = \text{Id}$, so Y intersects Π_- Bott-transversely (i.e. transversely in the normal direction to $\Pi_- \cap Y$) and hence $B = \Pi_- \cap Y$ is reduced. Consequently, Lemma 1.13 follows from the strong Bertini Theorem 5.1. (The case when the signs symbols $+$ and $-$ are interchanged is analogous.) \square

We now return to divisors in Grassmannians and prove Theorem 1.2. Let $Gr(k, n) \subset \mathbb{P}^N$ be the Plücker embedding, then the anti-canonical class of $Gr(k, n)$ equals $\mathcal{O}_{\mathbb{P}^N}(n)|_{Gr(k, n)}$ [23, Proposition 1.9]. Consequently, a smooth divisor $X \subset Gr(k, n)$ in the linear system $\mathcal{O}_{\mathbb{P}^N}(d)|_{Gr(k, n)}$ satisfies the W^+ condition, see Definition 2.1, if and only if

$$d \leq n \quad \text{or} \quad d \geq k(n - k) + n - 2,$$

and is monotone (Fano) if and only if $d < n$.

Proof of Theorem 1.2. We have already mentioned this theorem is easy and essentially known when $k(n - k)$ is even. (The sphere $L \subset X$ is nontrivial in $H_n(X)$ by Lemmas 4.1 and 3.8. Then apply Corollary 3.13.) We will now prove the hard case when $k(n - k)$ is odd using the general Theorem 1.12. Denote $k = 2p + 1$, $n = 2q$.

Consider a linear involution on \mathbb{C}^{2q} with $q + l$ positive eigenvalues and $q - l$ negative eigenvalues for some l . It induces a non-degenerate involution ι on $Gr(2p + 1, 2q)$ whose fixed locus is

$$Gr(2p + 1, 2q)^\iota = \bigsqcup_{t=0}^{2p+1} Gr(t, q + l) \times Gr(2p + 1 - t, q - l).$$

This fixed locus consists of $(2p + 1)$ -planes that admit a frame in which t vectors lie in the positive eigenspace of the involution on \mathbb{C}^{2q} , and the remaining $2p + 1 - t$

vectors lie in the negative eigenspace. We compute:

$$(5.1) \quad \dim Gr(t, q + l) + \dim Gr(2p + 1 - t, q - l) - \frac{1}{2} \dim Gr(2p + 1, 2q) = \\ = -\frac{1}{2}(1 + 2p - 2t)(1 + 2p + 2l - 2t).$$

For this paragraph, set $l = 0$. Then the expression (5.1) is less than 0 for any $t \in \mathbb{Z}$. This means $\dim Gr(2p + 1, 2q)^\iota < \frac{1}{2} \dim Gr(2p + 1, 2q)$. (The left-hand side is disconnected, and we mean that the inequality holds for each of its connected components.) Therefore we can apply Lemma 1.13 to either of the two linear systems $\mathbb{P}H^0(Y, \mathcal{L}^{\otimes d})_\pm$. In order to apply Theorem 1.12, it remains to check that $Gr(2p + 1, 2q)^\iota$ contains a connected component of even dimension. A computation shows that the t 'th connected component of $Gr(2k + 1, 2n)^\iota$ has dimension of parity

$$\dim Gr(t, q) + \dim Gr(2p + 1 - t, q) = q - 1 \pmod{2}$$

independently of t . These dimensions are even when q is odd; for now assume this is the case. If d is odd, apply Theorem 1.12(b) taking either of the two sign symbols $+$ or $-$. If d is even, apply Theorem 1.12(a) (this case is easier and does not require the computation of dimensions we've made). We have proved Theorem 1.2 for $Gr(2p + 1, 2q)$ when q is odd.

Now suppose q is even. Set $l = 1$ till the end of the proof. Recall that $Gr(2p + 1, 2q)^\iota = (\Pi_+ \sqcup \Pi_-) \cap Gr(2p + 1, 2q)$. The only case when (5.1) fails to be less than zero is when

$$1 + 2p - 2t = -1.$$

This happens for a unique $t \in \mathbb{Z}$. So either $\dim Gr(2p + 1, 2q) \cap \Pi_+ < \frac{1}{2} \dim Gr(2p + 1, 2q)$, or the same holds with Π_- taken instead. (As above, we mean that the inequality holds for each connected component of the left hand side.) A computation shows that the t 'th connected component of $Gr(2p + 1, 2q)^\iota$ has dimension of parity

$$\dim Gr(t, q + 1) + \dim Gr(2p + 1 - t, q - 1) = q \pmod{2} = 0$$

Therefore we can apply Lemma 1.13 and Theorem 1.12 taking that symbol $+$ or $-$ for which the inequality $\dim Gr(2p + 1, 2q) \cap \Pi_\mp < \frac{1}{2} \dim Gr(2p + 1, 2q)$ holds. Theorem 1.2 is proved in all cases. \square

Proof of Corollaries 1.3, 1.14. These corollaries follow from Theorems 1.2, 1.12 and Lemma 3.14. \square

APPENDIX A. INVOLUTIONS, GROWTH OF LAGRANGIAN FLOER HOMOLOGY AND RING STRUCTURES

A.1. Growth of Lagrangian Floer homology and ring structures. Ailsa Keating [17] has recently obtained an exact sequence involving iterated Dehn twists in the Fukaya category of a symplectic manifold (extending Paul Seidel's original exact sequence [35]). In this subsection we use it to prove Proposition A.1 (stated below). Then we apply it to compute Floer homology rings of vanishing spheres in some divisors.

Let X be a compact monotone symplectic manifold. Denote by $\mathcal{F}(X) = \bigoplus_{\lambda \in \mathbb{R}} \mathcal{F}(X)_\lambda$ its monotone Fukaya category over \mathbb{C} , decomposed into summands corresponding to the eigenvalues of multiplication with $c_1(X)$ in $QH(X)$. The basic language of A_∞ and Fukaya categories is explained in [36], and the monotone version of the Fukaya category is discussed in [39]. Our aim is to prove the following.

Proposition A.1. *Let X be a monotone symplectic manifold, $\dim_{\mathbb{R}} X = 4k$ for some $k \geq 1$, $L_1 \subset X$ a Lagrangian sphere and $L_2 \subset X$ another monotone Lagrangian which intersects L_1 once transversely. Assume L_1, L_2 are included into the same summand $\mathcal{F}(X)_\lambda$. Suppose that $\dim HF(\tau_{L_1}^k L_2, L_2)$ is unbounded with $k \in \mathbb{N}$. Then there is an isomorphism of rings $HF(L_1, L_1) \cong \mathbb{C}[x]/x^2$.*

We need to introduce some notation.

Definition A.2. Let A be a strictly unital $\mathbb{Z}/2$ -graded A_∞ algebra with unit $1 \in A$, M a right A_∞ module over A and N a left A_∞ module over A . Fix an augmentation, i.e. a vector space splitting $A = (1) \oplus \bar{A}$. The k -truncated bar complex is the vector space

$$(M \otimes_A N)_k := \bigoplus_{j=0}^{k-1} M \otimes \bar{A}^{\otimes j} \otimes N$$

with the differential that on the j 'th summand equals

$$\sum_{\substack{j+2=p+q+r, \\ p, r \geq 0}} (-1)^{\mathfrak{A}} (-1)^r (\text{Id}^{\otimes p} \otimes \mu^q \otimes \text{Id}^{\otimes r}).$$

Here $\mathfrak{A} \in \{0, 1\}$ depends on gradings of the arguments: if the input is $m \otimes x_1 \otimes \dots \otimes x_{k-1} \otimes n$, $m \in M$, $x_i \in A$, $n \in N$ then \mathfrak{A} is the mod 2 sum of gradings of the last r elements of the input. When $p = 0$, we get the summand $\mu^q \otimes \text{Id}^{\otimes r}$ which contains the module structure map $\mu^q : M \otimes A^{\otimes(q-1)} \rightarrow M$. Similarly, when $r = 0$, μ^q becomes the module structure map $\mu^q : A^{\otimes(q-1)} \otimes N \rightarrow N$. When $p, r > 0$, μ^q is the algebra structure map $A^{\otimes q} \rightarrow A$ composed with the augmentation $A \rightarrow \bar{A}$.

Theorem A.3 (Keating, [17, Lemma 7.2 and Remark 6.6]). *Suppose $L_1, L, L_2 \subset X$ are three Lagrangian submanifolds included in $\mathcal{F}(X)_\lambda$, L is a sphere. There is an exact sequence of vector spaces*

$$\begin{array}{ccc} HF(L_1, L_2) & \xrightarrow{\quad} & HF(\tau_L^k L_1, L_2) \\ & \nwarrow \quad \swarrow & \\ H \left(\text{Hom}(L, L_1) \otimes_{\text{Hom}(L, L)} \text{Hom}(L_2, L) \right)_k & & \end{array}$$

□

Note that [17] proves the theorem over $\mathbb{Z}/2$ and for exact X , but the same arguments work for monotone X , and keeping track of signs will prove the theorem over \mathbb{C} . Next we state some auxiliary lemmas.

Lemma A.4 (Formality). *Every A_∞ algebra whose cohomology ring is $\mathbb{C}[x]/(x^2 - 1)$ is quasi-isomorphic to the A_∞ algebra $\mathbb{C}[x]/(x^2 - 1)$ with vanishing higher multiplications: $\mu^j = 0, j > 2$.*

Proof. This follows from [16, Corollary 4] and [15, Proposition 2.2]. \square

Lemma A.5. *Take the A_∞ algebra $\mathbb{C}[x]/(x^2 - 1)$ with vanishing $\mu^j, j > 2$. Every strictly unital A_∞ module M over this algebra with vanishing μ^1 necessarily has vanishing $\mu^j, j > 2$.*

Proof. Take the minimal j such that $\mu^j(m, x^{\otimes j}) \neq 0$ for some $m \in M$. If $j > 2$, the A_∞ relation for the tuple $(m, x^{\otimes j}, 1)$ gives $\mu^j(m, x^{\otimes j}) = 0$, a contradiction. \square

Lemma A.6 ([17, Lemma 3.1]). *Let (M, A, N) be a c -unital A_∞ category consisting of an A_∞ algebra A , a left A_∞ module M and a right A_∞ module N . Let A' be a strictly unital A_∞ algebra quasi-isomorphic to A . Then there are strictly unital A_∞ modules M', N' over A' such that the category (M, A, N) is quasi-isomorphic to (M', A', N') .* \square

Lemma A.7 (Cf. [17, Lemma 7.3]). *Let (M, A, N) and (M', A', N') be two strictly unital A_∞ categories consisting of an algebra, a left module and a right module. If they are quasi-isomorphic, the associated bar complexes $(M \otimes_A N)_k$ and $(M' \otimes_{A'} N')_k$ are quasi-isomorphic.* \square

Remark A.8. Denote $\dim_{\mathbb{R}} X = 2n$. Suppose $L \subset X$ is a Lagrangian sphere. The $\mathbb{Z}/2$ -graded Floer chain complex $CF(L, L)$ can be realised as a 2-dimensional vector space $\mathbb{C} \oplus \mathbb{C}$ with two generators: the unit 1, $\deg 1 = 0$ and the second generator x , $\deg x = n \pmod 2$. The differential has degree 1.

If n is even, Floer's differential must vanish and $HF(L, L)$ is a unital 2-dimensional commutative algebra. Up to isomorphism, this leaves only two possibilities: $\mathbb{C}[x]/x^2$ or $\mathbb{C}[x]/(x^2 - 1)$.

If n is odd, $HF(L, L)$ is zero or 2-dimensional. In the latter case, $x^2 = 0$ because $HF(L, L)$ is graded commutative, so $HF(L, L) \cong \mathbb{C}[x]/x^2$.

The minimal Chern number of X is the maximal integer N such that $c_1(X)$ is divisible by N in integral cohomology $H^2(X; \mathbb{Z})$. Floer homology of a Lagrangian sphere can be made $\mathbb{Z}/2N$ graded, and the above generators have gradings $\deg 1 = 0$, $\deg x = n \pmod{2N}$. If n is even and $n \neq 0 \pmod N$, by grading reasons we get $x^2 = 0$ and $HF(L, L) \cong \mathbb{C}[x]/x^2$.

Proof of Proposition A.1. We want to prove that $HF(L_1, L_1) \cong \mathbb{C}[x]/x^2$. Suppose this is not the case, then by Remark A.8 $HF(L_1, L_1; \mathbb{C}) \cong \mathbb{C}[x]/(x^2 - 1)$, recall that n is even.

Take inside $\mathcal{F}(X)_\lambda$ the subcategory consisting of the A_∞ algebra $Hom(L_1, L_1)$, its left module $Hom(L_1, L_2)$ and its right module $Hom(L_2, L_1)$. Because $|L_1 \cap L_2| = 1$, $Hom(L_1, L_2)$ and $Hom(L_2, L_1)$ are 1-dimensional as vector spaces. Denote their generators by

$$Hom(L_1, L_2) = \langle m \rangle, \quad Hom(L_2, L_1) = \langle n \rangle.$$

By Lemma A.4 $Hom(L_1, L_1)$ is quasi-isomorphic to the associative algebra $\mathbb{C}[x]/(x^2 - 1)$ with trivial higher multiplications. By Lemma A.6 and Lemma A.5, modules

$\text{Hom}(L_1, L_2)$ and $\text{Hom}(L_2, L_1)$ are quasi-isomorphic to those with trivial higher multiplications over $\mathbb{C}[x]/(x^2 - 1)$. The μ^2 -operations, however, must be nontrivial because $x^2 = 1$:

$$\mu^2(m, x) = \epsilon_m m, \quad \mu^2(x, n) = \epsilon_n n \quad \text{where} \quad \epsilon_m, \epsilon_n = \pm 1.$$

Lemma A.7 allows to compute homology of the bar complex

$$B_k := (\text{Hom}(L_1, L_2) \otimes_{\text{Hom}(L_1, L_1)} \text{Hom}(L_2, L_1))_k$$

using the simple associative model we obtained. In this model, the bar complex is based on the k -dimensional vector space

$$\bigoplus_{j=1}^{k-1} m \otimes x^{\otimes j} \otimes n.$$

The differential comes only from $\mu^2(m, x)$ and $\mu^2(x, n)$:

$$\partial(m \otimes x^{\otimes j} \otimes n) = ((-1)^j \epsilon_n + \epsilon_m) m \otimes x^{\otimes(j-1)} \otimes n.$$

Note that $(-1)^{\mathfrak{X}} = 1$ because we are given $\deg x = 0$ and may assume $\deg n = 0$. We see that $\dim H(B_k) = 0$ or 1 , depending on the parity of k . By the exact sequence of Theorem A.3, we get $\dim HF(\tau_{L_1}^k L_2, L_2; \mathbb{C}) \leq 2$, which contradicts to the hypothesis. \square

Remark A.9. If $HF(L_1, L_1; \mathbb{C}) \cong \mathbb{C}[x]/x^2$, it might still happen that $\text{Hom}(L_1, L_1)$ is formal, for example when X is exact. Running the above proof, from $x^2 = 0$ we conclude that $\mu^2(m, x) = \mu^2(x, n) = 0$. So the differential on the k -dimensional model for B_k written above vanishes, and $\dim H(B_k) = k$. This agrees with the growth of $\dim HF(\tau_{L_1}^k L_2, L_2)$.

A.2. Floer homology rings of Lagrangian spheres in divisors. We now combine Proposition A.1 with previous results (Propositions 1.10 and 4.2) to compute the ring $HF(L, L; \mathbb{C})$ for vanishing Lagrangian spheres L in certain divisors. The statement uses notation from Subsection 1.7. We also provide a corollary which specialises to divisors in Grassmannians.

Proposition A.10. *In addition to conditions of Theorem 1.12 (a) or (b), suppose X is Fano and $\dim_{\mathbb{C}} X$ is even. Then there is a ring isomorphism $HF(L, L; \mathbb{C}) \cong \mathbb{C}[x]/x^2$.*

Corollary A.11. *Let $X \subset Gr(k, n)$ be a smooth divisor of degree $3 \leq d < n$, $\dim_{\mathbb{C}} X$ even. Let $L \subset X$ be an $|\mathcal{O}(d)|$ -vanishing Lagrangian sphere. Then there is a ring isomorphism $HF(L, L; \mathbb{C}) \cong \mathbb{C}[x]/x^2$.*

The possibility ruled out by these two statements is the deformed ring $HF(L, L) \cong \mathbb{C}[x]/(x^2 - 1)$. An example of a sphere with $HF(L, L) \cong \mathbb{C}[x]/(x^2 - 1)$ is the antidiagonal $L \subset \mathbb{P}^1 \times \mathbb{P}^1$. Note that for this sphere, τ_L has order 2 in $\pi_0 \text{Sym}(\mathbb{P}^1 \times \mathbb{P}^1)$ [37]. It seems natural to ask whether there is a general relation between the isomorphism $HF(L, L) \cong \mathbb{C}[x]/(x^2 - 1)$ and τ_L being of finite order (both cases are rare).

For $X \subset \mathbb{P}^3$ the cubic surface, Corollary A.11 has recently been proved by Nick Sheridan [39]. That proof could perhaps be generalised to prove Corollary A.11 for

all projective hypersurfaces, because [39] relies on the knowledge of $QH^*(X)$ which is known for all hypersurfaces. For many, but not all, pairs (k, n) Corollary A.11 follows from grading considerations, see Remark A.8.

Proof of Proposition A.10. As in the beginning of the proof of Theorem 1.12, take X, L_1, L_2 coming from Proposition 4.2. By Lemma 3.8, it suffices to prove that $HF(L_1, L_1) \cong \mathbb{C}[x]/x^2$.

From the Picard-Lefschetz formula (Lemma 3.12), given that $|L_1 \cap L_2| = 1$ and $\dim L_i^\iota$ is odd, we get the equality $[\tau_{L_1^\iota}^k L_2^\iota] = [L_2^\iota] - \epsilon k [L_1^\iota]$ in the homology of the fixed locus $H_*(X^\iota)$. Consequently, $[\tau_{L_1^\iota}^k L_2^\iota] \cdot [L_2^\iota] = -\epsilon k$. By Proposition 1.10, $\dim HF(\tau_{L_1}^k L_2, L_2) \geq k$. By Proposition A.1, $HF(L_1, L_1) \cong \mathbb{C}[x]/x^2$. \square

Proof of Corollary A.11. Repeat the proof of Theorem 1.2 but refer to Proposition A.10 instead of Theorem 1.12. Recall the condition $d < n$ means that X is Fano. \square

REFERENCES

- [1] M. Atiyah. On analytic surfaces with double points. *Proc. Royal Soc. Ser. A*, 247:237–244, 1958.
- [2] R. Avdek. Liouville hypersurfaces and connected sum cobordisms. *arXiv:1204.3145 [math.SG]*, 2012.
- [3] M. C. Beltrametti, M. L. Fania, and A. J. Sommese. On the discriminant variety of a projective manifold. *Forum Mathematicum*, 4:529–547, 1992.
- [4] D. Ben-Zvi and D. Nadler. Secondary Traces. *arXiv:1305.7177 [math.AG]*, 2013.
- [5] M. Betz and J. Rade. Products and relations in symplectic Floer homology. *arXiv:dg-ga/9501002*, 1995.
- [6] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder. Compactness results in Symplectic Field Theory. *Geometry and Topology*, 7:799–888, 2003.
- [7] F. Bourgeois and K. Mohnke. Coherent orientations in symplectic field theory. *Mathematische Zeitschrift*, 248(1):123–146, 2004.
- [8] M. J. Cowen. Automorphisms of Grassmannians. *Proceedings of the American Mathematical Society*, 106(1):99–106, 1989.
- [9] S. Diaz and D. Harbater. Strong Bertini Theorems. *Transactions of the American Mathematical Society*, 324(1):73–86, 1991.
- [10] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to Symplectic Field Theory. In *Visions in Mathematics*, pages 560–673. Birkhäuser, 2010.
- [11] A. Floer and H. Hofer. Coherent orientations for periodic orbit problems in symplectic geometry. *Mathematische Zeitschrift*, 212(1):13–38, 1993.
- [12] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. *Lagrangian Intersection Floer Theory: Anomaly and Obstruction*, volume 46 of *Studies in Advanced mathematics*. American Mathematical Society, International Press.
- [13] K. Fukaya and K. Ono. Arnold conjecture and Gromov-Witten invariant. *Topology*, 38(5):933–1048, 1999.
- [14] H. Hofer and D. A. Salamon. Floer homology and Novikov rings. In H. Hofer, C. Taubes, A. Weinstein, and E. Zehnder, editors, *The Floer Memorial Volume*, pages 483–524. Birkhäuser, 1995.
- [15] T. Holm. Hochschild cohomology rings of algebras $k[X]/(f)$. *Beiträge Algebra Geom.*, 41:291–301, 2000.
- [16] T. V. Kadeishvili. The structure of the A_∞ -algebra, and the Hochschild and Harrison cohomologies. *Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruz. SSR*, 91:19–27, 1988.

- [17] A. Keating. Dehn twists and free subgroups of symplectic mapping class groups. *arXiv:1204.2851 [math.SG]*, 2012.
- [18] M. Khovanov and P. Seidel. Quivers, Floer cohomology, and braid group actions. *J. Amer. Math. Soc.*, 15:203–271, 2002.
- [19] N. A. Krylov. Relative Mapping Class Group of the Trivial and the Tangent Disk Bundles over the Sphere. *Pure and Applied Mathematics Quarterly*, 3(3):631–645, 2007.
- [20] M. Lönne. Fundamental groups of projective discriminant complements. *Duke Math. J.*, 152(2):357–405, 2009.
- [21] E. J. N. Looijenga. Cohomology and intersection homology of algebraic varieties. volume 2 of *Park City IAS Mathematics Series*, pages 221–263. AMS, 1992.
- [22] D. McDuff and D. A. Salamon. *J-Holomorphic Curves and Symplectic Topology*, volume 52 of *AMS Colloquium Publications*. 2004.
- [23] S. Mukai. Curves and Grassmannians. In *Algebraic Geometry and Related Topics (Proceedings of the International Symposium, Incheon, Republic of Korea, February 11-13, 1992)*, pages 19–40. International Press, 1993.
- [24] Y.-G. Oh. Floer cohomology of Lagrangian intersections and pseudo-holomorphic disks I. *Communications on Pure and Applied Mathematics*, 46(7):949–993, 1993.
- [25] A. Polishchuk. Lefschetz type formulas for dg-categories. *Selecta Mathematica*, pages 1–44, 2013.
- [26] A. Ritter and I. Smith. The open-closed string map revisited. *arXiv:1201.5880 [math.SG]*, 2012.
- [27] D. A. Salamon and S. Dostoglou. Self-dual instantons and holomorphic curves. *Annals of Mathematics*, 139:581–640, 1994.
- [28] D.A. Salamon and E. Zehnder. Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. *Comm. Pure Appl. Math.*, 45:1303–1360, 1992.
- [29] M. Schwarz. *Morse Homology*, volume 111 of *Progress in Mathematics*. 1993.
- [30] M. Schwarz. *Cohomology Operations from S^1 -Cobordisms in Floer Homology*. PhD thesis, 1995.
- [31] P. Seidel. Abstract analogues of flux as symplectic invariants. *arXiv:1108.0394 [math.SG]*.
- [32] P. Seidel. *Floer homology and the symplectic isotopy problem*. PhD thesis, 1997.
- [33] P. Seidel. π_1 of symplectic automorphism groups and invertibles in quantum homology rings of symplectic automorphism groups and invertibles in quantum homology rings. *Geom. Funct. Anal.*, 7:1046–1095, 1997.
- [34] P. Seidel. Graded Lagrangian submanifolds. *Bulletin de la Société Mathématique de France*, 128(1):103–149, 2000.
- [35] P. Seidel. A long exact sequence for symplectic Floer cohomology. *Topology*, 42:1003–1063, 2003.
- [36] P. Seidel. *Fukaya Categories and Picard-Lefschetz Theory*. European Mathematical Society, Zurich, 2008.
- [37] P. Seidel. Lectures on Four-Dimensional Dehn Twists. In *Symplectic 4-Manifolds and Algebraic Surfaces*, volume 1938 of *Lecture Notes in Mathematics*, pages 231–267. Springer, 2008.
- [38] P. Seidel and I. Smith. Localization for Involutions in Floer Cohomology. *Geometric and Functional Analysis*, 20(6):1464–1501, 2010.
- [39] N. Sheridan. On the Fukaya category of a Fano hypersurface in projective space. *arXiv:1306.4143 [math.SG]*, 2013.
- [40] I. Smith. Floer cohomology and pencils of quadrics. *Inventiones mathematicae*, 189(1):149–250, 2012.
- [41] E. A. Tevelev. *Projective Duality and Homogeneous Spaces*, volume 133 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, 2005.
- [42] K. Wehrheim and C. Woodward. Exact triangle for fibered Dehn twists. *Preprint*.
- [43] K. Wehrheim and C. Woodward. Functoriality for Lagrangian correspondences in Floer theory. *Quantum topology*, 1:129–170, 2010.
- [44] K. Wehrheim and C. Woodward. Quilted Floer cohomology. *Geometry and Topology*, 14:833–902, 2010.

DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE, CB3 0WB, UK